# An Integrated Interval Neural Network for Uncertainty Modeling in Inhomogeneous Materials

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**Abstract.** Engineering fields rely heavily on the Finite Element Method (FEM) as a modeling tool in deterministic systems where no uncertainty is introduced. The effects of uncertainty are of growing concern in the analysis and design of engineering structures and need to be studied to improve the predictability of mathematical models. Recently, in addition to others, Interval Finite Element Method (IFEM) has been introduced to account for uncertainties by incorporating interval arithmetic into the conventional FEM formulation, in which all uncertain parameters are defined as intervals. Nonetheless, in combination with complexity of structures and inhomogeneous materials, the computational and experimental cost remains an inevitable issue in such simulations.

This work aims at integrating Artificial Neural Networks (ANN) and IFEM techniques to establish a flexible and efficient approach for modeling uncertainties in general inhomogeneous structures, in which an Interval Neural Network (INN) is employed as a substitution for the conventional constitutive material model to establish a homogenized representation of structures regardless of material complexity. In this approach, at first, the required dataset is generated by creating and running a set of IFEM simulations. The INN will then be trained to predict the homogenized mechanical behavior of the structure as a function of independent parameters. Afterwards, the trained INN will be integrated in the IFEM procedure to obtain the system's response under uncertainty. The proposed approach is applied to a set of engineering problems to illustrate and verify the capabilities of the methodology.

**Keywords:** Interval Finite Element Method, Interval Neural Network, Machine Learning, Inhomogeneous Material

## 1. Introduction

The effects of uncertainty are of growing concern in the analysis and design of engineering structures. In general, uncertainties play an important role in the modeling of various engineering and science problems and need to be studied for improving the predictability of mathematical models. Among various proposed approaches for modeling uncertainty, interval-valued data are a natural way to represent uncertainty in engineering systems, e.g. uncertainty in measurements that contain valuable information useful for decision making.

Interval Finite Element Method (IFEM) is an attempt to account for uncertainty in structural analysis by incorporating interval arithmetic into conventional finite element formulation (Muhanna and Mullen, 2001). Within the context of IFEM, all uncertain parameters, such as applied loads, material properties and geometry, are defined as intervals (Rao et al., 2011; Xiao et al., 2015; Muhanna and Shahi, 2020). The main challenge in IFEM is reducing the overestimation due to interval dependency problem which is handled by decomposing matrices into interval and deterministic components, applying Element-By-Element assembly approach and employing iterative enclosure method to solve the system of equations (Muhanna and Mullen, 2001; Rao et al., 2011; Xiao, 2015).

One important component which is an integral part in computational mechanics analysis including FEM and IFEM is constitutive material model. In general, constitutive models describe the material responses to various mechanical conditions and represent the stress-strain relations (Bower, 2009). Such descriptions are mainly presented as mathematical models which are a simplification of complex mechanical behaviors studied through experiments. The level of simplification depends on the purpose and the required precision of the model predictions. For instance, the mechanical behavior of steel can be represented by an elastic-plastic model but to analyze structures under working loads, engineers usually employ linear elasticity model in which Hooke's law represents the material behavior assuming the strain proportional to the applied stress (Bower, 2009; Kenneth et al., 2006).

The conventional process of developing material models mainly includes developing the mathematical formulation of constitutive equations using principles of mathematics and continuum mechanics, and determining the associated parameters to fit experimental measurements (Desai and Siriwardane, 1984). However, establishing an exact closed form constitutive model is not possible for all materials and the requirement for a sufficiently accurate model can lead to obtaining complex and computationally expensive models. Especially, with the introduction and application of composite and inhomogeneous materials in general, providing constitutive models describing complex mechanical behaviors becomes more challenging and there will be an ever-increasing demand for more realistic models (Buyukozturk and Shareef, 1985; Ghaboussi et al., 1999).

Artificial neural networks (ANN) are computational tools inspired by human perception of the biological structure of neurons and the internal operation of brain. A neural network is a nonlinear system usually consisting of a large number of highly interconnected processing units known as artificial neurons and can provide models representing highly nonlinear behaviors. Feedforward supervised neural networks are known as very first and successful learning algorithms (Rumelhart et al., 1985). The universal approximation theorem (Cybenko, 1989; Hornik et al., 1989) guarantees that with even a single hidden layer containing a finite number of neurons, this type of networks can represent arbitrary complex but smooth approximation of any arbitrary input-output mapping (White, 1990; Goodfellow et al., 2016). Such great flexibility in learning nonlinear relationships between input and output parameters makes ANN a robust alternative to mathematical constitutive models, in particular where the complexity of material response to applied mechanical stimuli cannot be easily and accurately described by mathematical models.

Recently, a variety of ANN constitutive models have been proposed to represent complex mechanical behaviors of various materials (Ghaboussi et al., 1999; Sen et al., 2002; Jung and Ghaboussi, 2006) and have also been successfully integrated in FEM procedure (Javadi et al., 2003; Gulikers, 2018). To the best of our knowledge, all the proposed models are deterministic in which the effects of uncertainty have not been taken into account and the input and output of the associated neural networks are crisp (i.e. single-valued) data (Ghaboussi et al., 1999; Sen et al., 2002; Jung and Ghaboussi, 2006).

In general, an ANN is considered INN if at least one of its input, output, or assigned weight sets are intervals (Beheshti et al., 1998). Accordingly, a variety of INNs are proposed. One of the very first version of INNs presented by Ishibuchi et al. in which except the input data, all other numbers are interval-valued and applied for fuzzy regression analysis (Ishibuchi et al., 1993). The universal approximation theorem is also proven for this type of INN (Baker and Patil, 1998). Repeating interval operations in INN with interval-valued inputs, outputs, and weights cause the explosion of interval uncertainty in these networks (Simoff, 1996). One straight-forward method proposed by Rossi and Conan-Guez is the extremal approach in which each interval-valued input is transformed into a pair of real numbers, for instance the corresponding lower bound and upper bound, and the INN is built on top of a standard Multi-layer Perceptron (MLP) (Rossi and Conan-Guez, 2002). Another approach to handle the interval-valued data is interval Multi-layer Perceptrone (iMLP) proposed by San Roque et al. in which inputs and outputs are interval-valued data, but the weights and biases are single-valued data. In this method, the input and output of each neuron are a pair of center-range values and each center and range share the same weight (San Roque et al., 2007). Lately, a regularized artificial neural network (RANN) is proposed by Yang et al. in which a non-crossing regularizer is introduced to control the interval crossing problem (Yang et al., 2019). In this paper, a variant of extremal approach is employed to construct an INN constitutive material model.

This work is an effort to take advantage of interval neural networks (INN) trained by interval-valued data to model the uncertainty, for instance, in experimental measurements of applied stress and strain. Then, the trained INN is employed as a substitution for the conventional constitutive model (mathematical model) to represent a homogenized material model incorporated into IFEM procedure, and establish an integrated approach that accounts for uncertainty in analyzing structures with particularly inhomogeneous materials.

The paper is structured as follows. First, the key features of the extremal approach are briefly presented to handle the interval-valued inputs and outputs data. The INN constitutive model and the integrated framework are explained in the next sections providing the details regarding the substitution of conventional constitutive model with the trained INN material model. Finally, numerical examples are presented and discussed to verify and demonstrate the application of proposed approach.

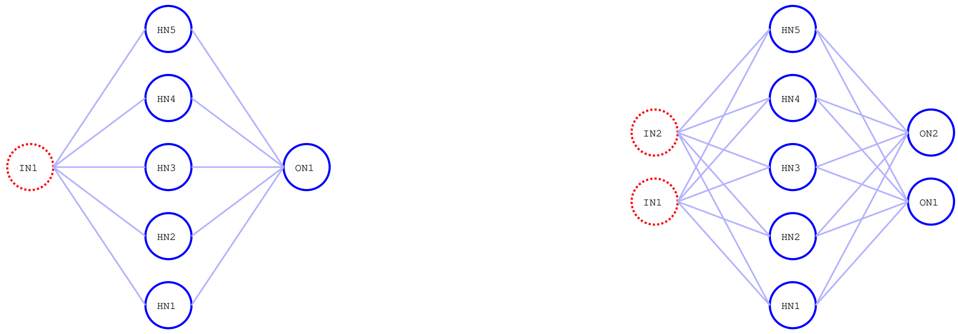



*Figure 1.* Shallow artificial neural networks. Single input single output network (left). Modified architecture to handle the interval-valued data in extremal approach (right).

## 2. Formulation

### 2.1. Extremal Approach

This approach is most probably the simplest approach to deal with interval-valued input and can be easily built on top of regular MLP. Each interval-valued data is defined by a pair of numbers: a lower bound and an upper bound, or a center value and a radius. A natural way for handling interval inputs is treating them as pair of inputs. In this way, instead of training the network with $n$ interval-valued input and output data, $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n}$ and $\boldsymbol{y_1}, \boldsymbol{y_2}, \ldots, \boldsymbol{y_n}$, where $\boldsymbol{x_i} = [\underline{x_i}, \overline{x_i}]$ and $\boldsymbol{y_i} = [\underline{y_i}, \overline{y_i}]$, respectively, $2n$ single-valued numbers $\underline{x_1}, \overline{x_1}, \underline{x_2}, \overline{x_2}, \ldots, \underline{x_n}, \overline{x_n}$ and $\underline{y_1}, \overline{y_1}, \underline{y_2}, \overline{y_2}, \ldots, \underline{y_n}, \overline{y_n}$ are used to train the neural network. Therefore, a single input single output neural network needs to be transformed to a neural network with two input and output neurons to be trained with a set of augmented data (Figure 1). In this work, each interval is considered as a pair of center and radius values denoted by $\boldsymbol{x_i} =< x_i^c, x_i^r >$. Thus, the augmented inputs are as follows:

$$x_1^c, x_2^c, \ \ldots, x_n^c, \tag{1}$$
$$x_1^r, x_2^r, \ \ \ldots, x_n^r \tag{2}$$

Then, the neural network is trained as a regular MLP with two input and output neurons (Figure 1 (right)).

### 2.2. INN Constitutive Model

Conventionally, to construct a material model, principles of mathematics and continuum mechanics suggest the form of mathematical equations. The associated parameters are then determined to fit the experimental stress and strain measurements obtained by standard mechanical tests, such as tensile test in which a tensile force is applied to a standard specimen of a material (homogeneous or inhomogeneous) and the corresponding displacement or strain response is measured (Kenneth et al., 2006). However, these measurements are not deterministic and the uncertainty in these data can naturally be presented as intervals. Such interval-valued data can be used to train an INN where the interval-valued stress is the input data, and the interval-valued module of elasticity is the output of the network.

In this research, the extremal approach explained in Section 2.1 is employed to construct such an INN in which pairs of center and radius values of interval-valued stresses is fed into the input layer and the associated pairs of center and radius values of module of elasticity are used as target values to train the INN. A regular backpropagation algorithm with Bayesian regularization is employed to train the INN and obtain the weights and biases. Such an INN represents the relationship between interval stress and strain and thus can effectively serve as the constitutive material model integrated into an IFEM analysis. Therefore, in this paper, such material descriptions are referred to as INN constitutive models, as opposed to the mathematical constitutive models which are known as conventional material models.

**Algorithm 1** Integrated Framework

**Input:** Problem definition (including geometry, loading , boundary conditions), the trained INN constitutive model
**Output:** System response (nodal displacements, internal forces, reaction forces, stresses, and strains)

```
for (step = 1 to load_steps) do
    Generating all possible endpoints combinations for loads and material properties
    for (i ∈ load_combinations) do
      Forming the global vector of equivalent nodal load for the current load step
      for (j ∈ material_properties_combinations (obtained by INN predictions)) do
        for (k = 1 to number_of_elements) do
          Forming the element i stiffness matrix
        end for
        Assembling the global stiffness matrix
        Imposing boundary conditions
        Solving the system of equations and obtaining primary unknowns (nodal displacements)
        Calculating secondary variables (stress and strains)
        Calculating principal stresses
        Storing the calculated values for the current combination of loads and material properties
      end for
    end for
    Report the minimum and maximum of the calculated values as the lower bound and the upper
    bound of the system response in the current load step
    Predicting the interval-valued material properties for the next load step by inputting the vector
    of interval-valued principal stresses into the trained interval neural network
end for
return The interval-valued results of the system response for the last load step
```

### 2.3. Integrated Framework

The proposed INN constitutive model captures the mechanical behavior of material and provides a description of material properties only based on the realistic experimental data without any prior knowledge about the material. Therefore, the integration of such model is straight forward, and it can be directly employed as conventional material models. in this paper, to obtain the exact enclosure, the proposed INN constitutive model is integrated in an endpoint combinations analysis in which the structure will be deterministically analyzed for all possible combinations of the lower bounds and upper bounds of interval parameters, e.g., applied loads and material properties. This method is basically applicable to the structures with a few interval parameters, since the number of combinations increases exponentially as the number of interval parameters increases. However, it is straight-forward to be implemented and a reliable method to validate the proposed approach. Algorithm 1 demonstrates the integrated procedure in which a trained INN-based description of material properties is directly used to form element stiffness matrices in the FEM endpoint combination analysis. The same effort can be made to integrate the INN constitutive model into IFEM procedure.

## 3. Example Problems

The proposed approach is implemented in a MATLAB program in which the INN is built on top of the MATLAB neural network toolbox. In this section, three numerical examples are presented to illustrate the application of the proposed framework. The first example presents a plane stress analysis of a cantilever beam with a linear elastic homogeneous material under uncertainty. Then, the second example illustrates the same cantilever beam built of an inhomogeneous material. Finally, the application of the proposed framework is further investigated by analyzing a plate built of the same inhomogeneous material in the third example. In this paper, to train the INN constitutive model, a set of synthetic measurements is generated by simulating a uniaxial tensile test of steel material, where the dimensions are chosen due to the standard size of specimens in uniaxial tension tests (Kenneth et al., 2006).

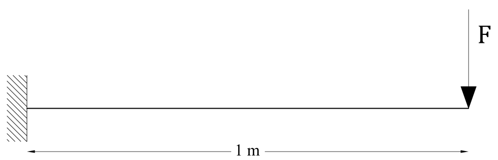



*Figure 2.* A cantilever beam subjected to a load at the free end.

### 3.1. Example 1: Plane Stress Analysis of a Cantilever Beam

The cantilever beam shown in Figure 2 is subjected to a vertical surface load $F = 100$ KN/m$^2$ at its free end. The length and the cross-sectional area are $L = 1$ m and $A = 0.01 \times 0.01$ m$^2$, respectively. The synthetic training dataset is generated by simulating a uniaxial tensile test of steel material (Figure 3), where the module of elasticity is set to $200 \times 10^6$ KN/m$^2$. To generate reasonable interval-valued data for stress and strain, an interval tensile load $\boldsymbol{P}$ is applied due to 1% uncertainty. The dataset is created by applying various interval tensile load $\boldsymbol{P}$ and obtaining the strain response of the specimen in an endpoint combination analysis in which the lower and upper bounds of the interval-valued data are determined by minimum and maximum values of the mechanical responses, respectively. Accordingly, a dataset including 400 datapoints is obtained, where 80 percent of them are randomly assigned to the train set and the rest 20 percent form the test dataset which is used to validate the trained INN.

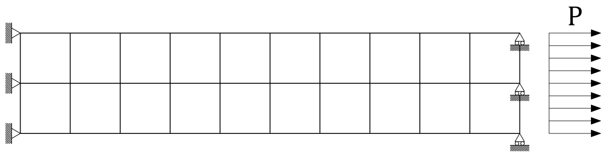



*Figure 3.* Tensile test simulation for generating synthetic data.

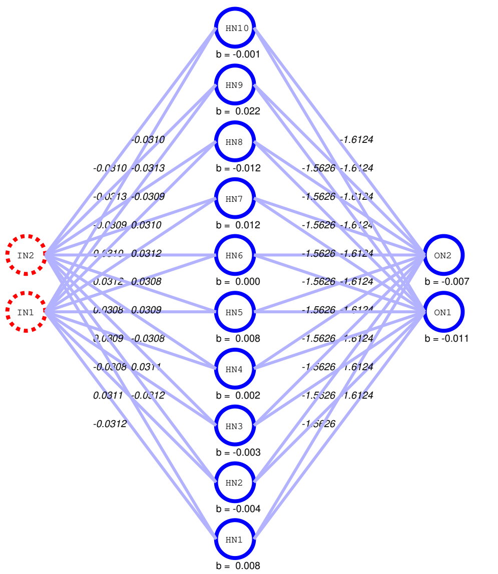



*Figure 4.* The trained interval neural network with ten neurons on the hidden layer.

The interval-valued stress-strain data is used to train a shallow feedforward INN with ten neurons on the hidden layer by employing the extremal approach explained in section 2.1. The inputs and outputs of this INN are pairs of mid points and radii of the interval-valued stress and strain data, respectively. Figure 4 illustrates the trained network using the synthetic data obtained by simulating a uniaxial tensile test on the homogeneous steel material.

Figure 5 demonstrates the linear relationship between the midpoints of stress and strain interval-valued data. The stress-strain relationship predicted by the trained INN shows that the neural network has successfully captured the material behavior. This INN is employed as a constitutive material model in endpoint combination analysis of the cantilever beam (Figure 2) which is modeled by ten second order quadrilateral elements illustrated in Figure 6.

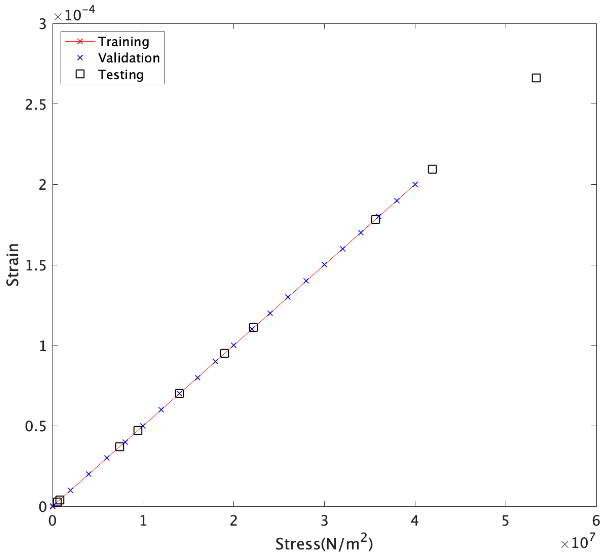



*Figure 5.* The midpoints of training, validation and test interval-valued data used for building the INN constitutive model.

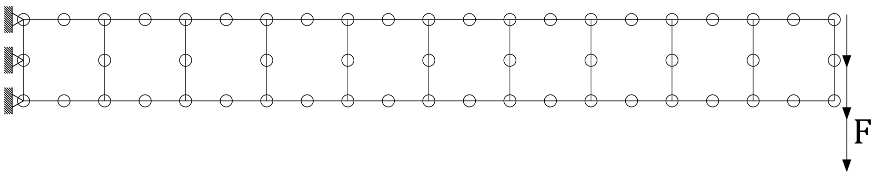



*Figure 6.* Modeling the cantilever beam with ten second-order quadrilateral elements

In this simulation, 1% uncertainty is considered in the applied surface load, and the material properties is predicted by employing the trained INN at each load step. Figure 7 demonstrates the deformed shape of the beam obtained by the midpoint values of interval-valued nodal displacements of the endpoint combination analysis with (i) INN constitutive model (red lines) and (ii) the conventional constitutive material model introduced by interval-valued module of elasticity (blue shaded elements). The outputs show a perfect match between the results of two analyses.

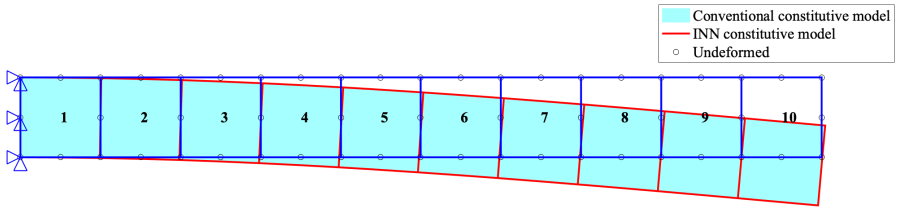



*Figure 7.* The cantilever beam with homogeneous material - deformed shape.

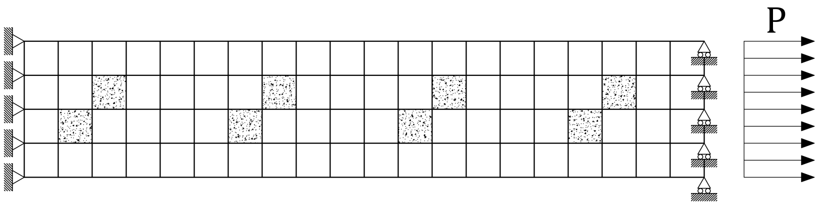



*Figure 8.* Tensile test simulation of an inhomogeneous material for generating synthetic data.

## 3.2. Example 2: Plane Stress Analysis of a Cantilever Beam with Inhomogeneous Material

To demonstrate the capability of the proposed approach in analyzing structures built of inhomogeneous materials, the same cantilever beam shown in Figure 2 is analyzed using the INN-based constitutive material model. The same extremal approach is adopted to obtain a homogenized INN constitutive material model of inhomogeneous materials under uncertainty. To train the INN, similar to the first example, a set of 400 datapoints are generated, where 80 percent of them randomly assigned to the train set and the rest 20 percent form the test set. Figure 8 demonstrates the simulation of the standard uniaxial tensile testing of an inhomogeneous material in which different material properties is assigned to some of the elements to model the inhomogenous material structure. Figure 9 illustrates that the stress-strain relationship is successfully captured by the homogenized INN constitutive model. It should be noted that in this example the homogenized material model demonstrates a linear form since the synthetic interval-valued stress and strain data are generated by a linear finite element analysis of a tensile test in which different modulus of elasticity are assigned to the elements. However, in general, the material can have any complicated nonlinear behavior, and due to the universal approximation theorem, the INN can capture such more complex stress-strain relations if enough data is provided for training.

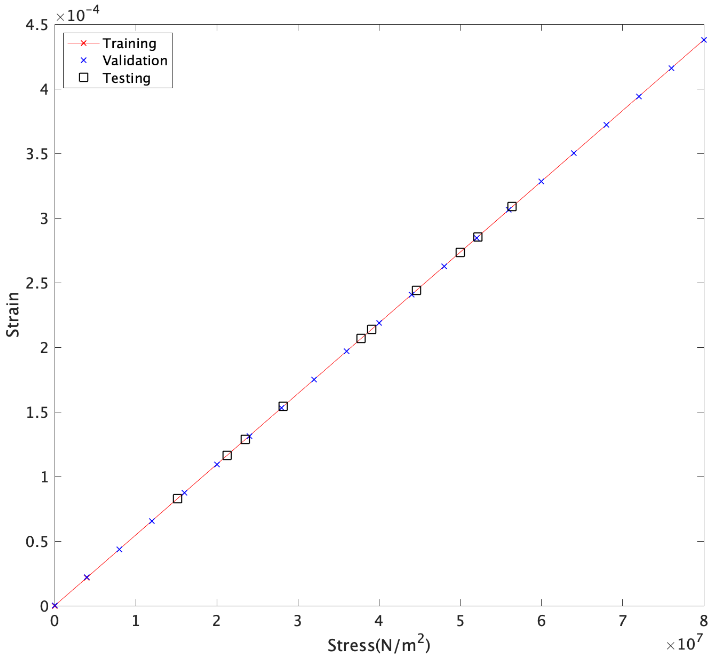



*Figure 9.* The midpoints of training, validation and test interval-valued data used for building interval neural network constitutive model.

The displacement response of the beam is illustrated in Figure 10. The lower and upper bounds of the horizontal and vertical displacements of eleven selected nodes are demonstrated in Figure 11 which shows the obtained intervals enclose the results of deterministic analysis. For instance, the deterministic numerical value of the vertical displacement at the tip of the cantilever beam (node 11) is $-0.0073$ mm which is enclosed by the interval $[-0.0075, -0.0072]$ mm obtained by the proposed framework.

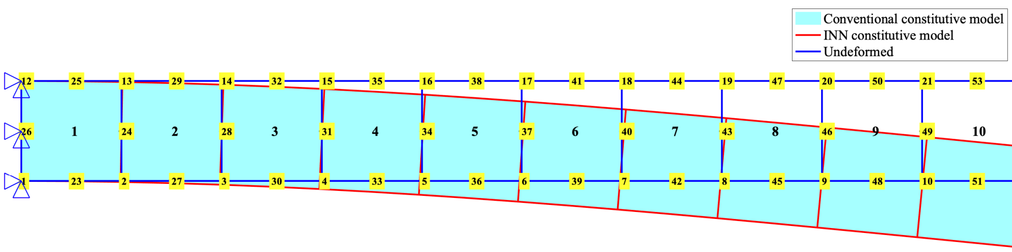



*Figure 10.* The cantilever beam with inhomogeneous material - deformed shape.

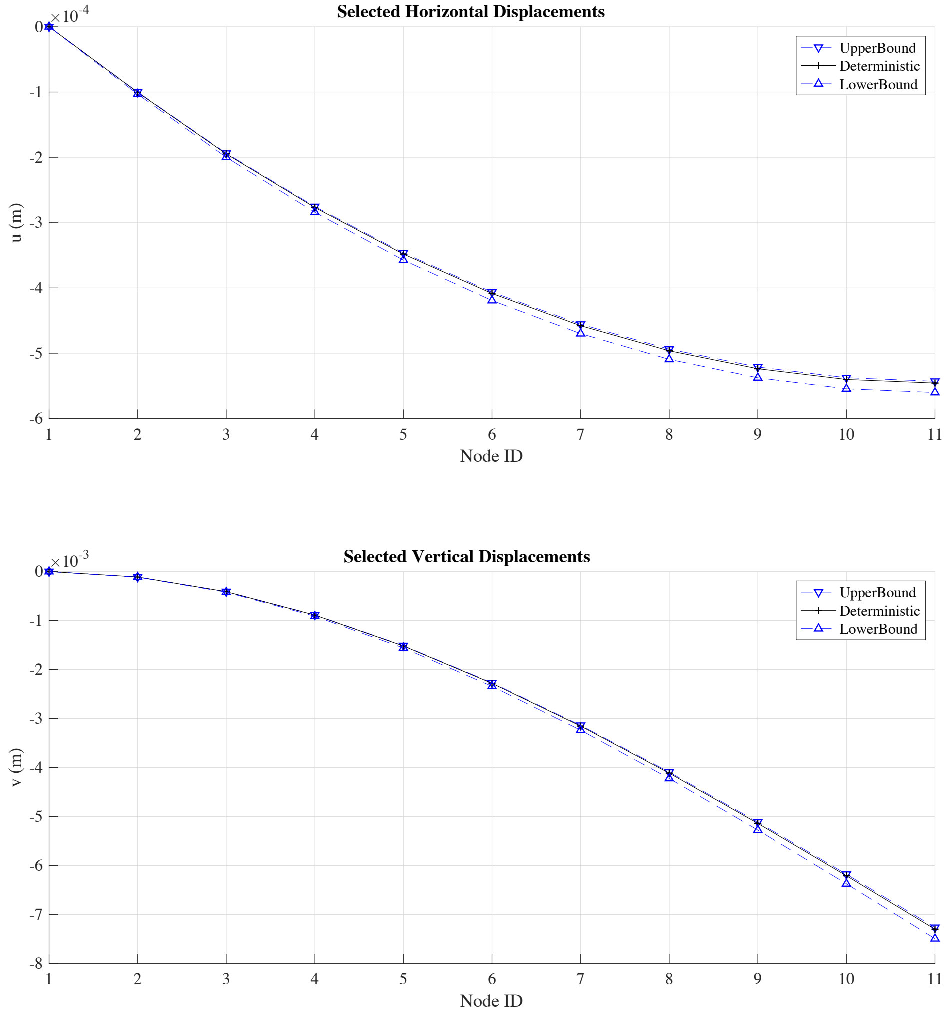



*Figure 11.* The selected displacement outputs.

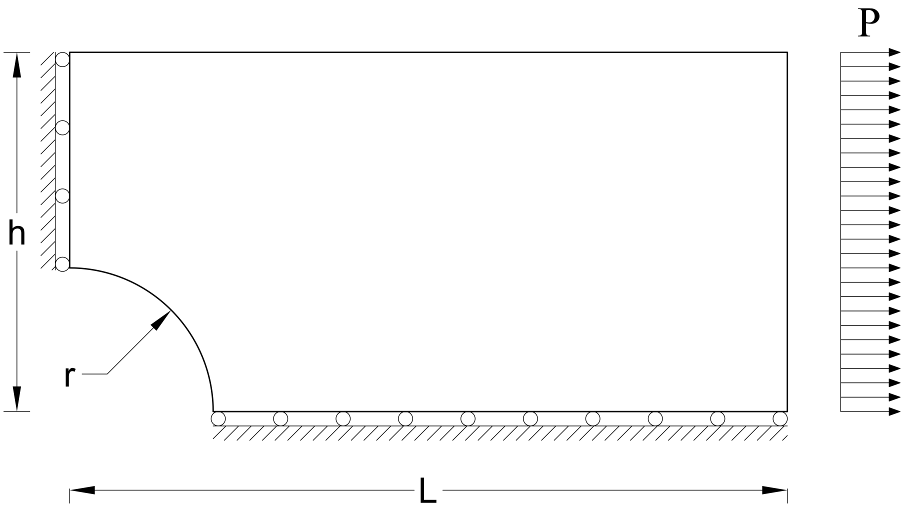



*Figure 12.* The rectangular plate with circular cutoff.

### 3.3. Example 3: Plane Stress Analysis of a Rectangular Plate with a Circular Cutoff Subjected to Uniformly Distributed Load

The same approach can be used for analyzing a rectangular plate with a circular cutoff shown in Figure 12. The dimension of this plate is specified by $h = 0.05$ m, $L = 0.1$ m, and $r = 0.02$ m. The thickness of the plate is $t = 0.005$ m and the right edge is subjected to a traction surface load of $P = [19.5, 20.5]$ N/mm$^2$. It is assumed that this plate is built of the same inhomogeneous material described in Example 2 except the synthetic measurement data is generated for 5% of uncertainty in the applied load. Therefore, it is expected that the interval width increases which is illustrated in Figure 13 where the upper and lower bounds are also added to the graph to demonstrate that the INN can capture the uncertainty in inhomogeneous material behavior. This neural network is trained with the same number and distribution of datapoints in train and test sets.

The deformed shape of the plate using the midpoints of interval displacement values are depicted in Figure 14, where the close match between the two deformed shapes obtained by the conventional and INN-based constitutive models exhibits the accuracy of the proposed approach. For more investigation, the displacements of the selected nodes on the edge of the plate are illustrated in Figure 15 in which the displacement responses obtained by the deterministic analysis when zero uncertainty is introduced in the model is fully enclosed by the bounds of the interval values calculated by the proposed approach.

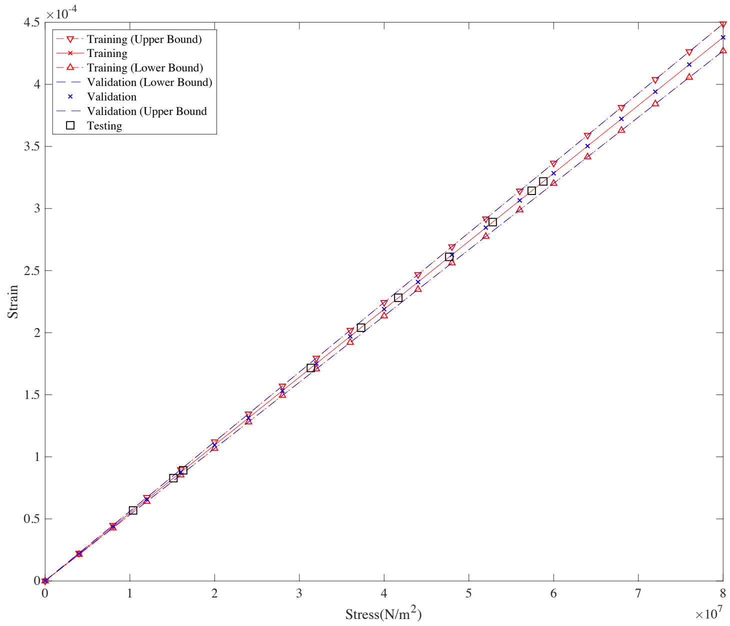



*Figure 13.* The training, validation, and test interval-valued data used for construction the interval neural network constitutive model. The training data is generated with 5% uncertainty level.

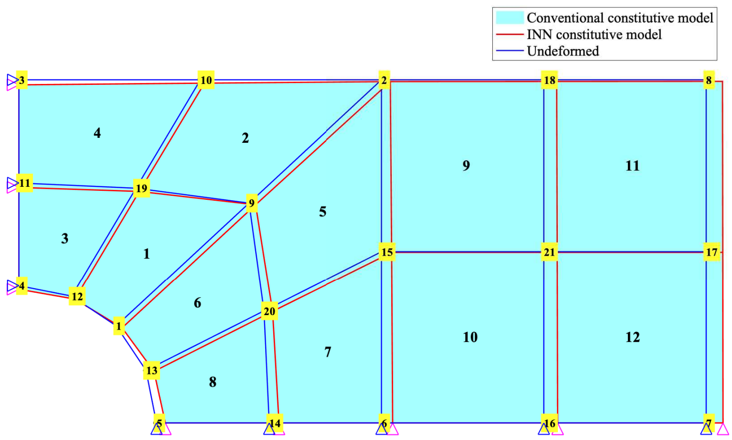
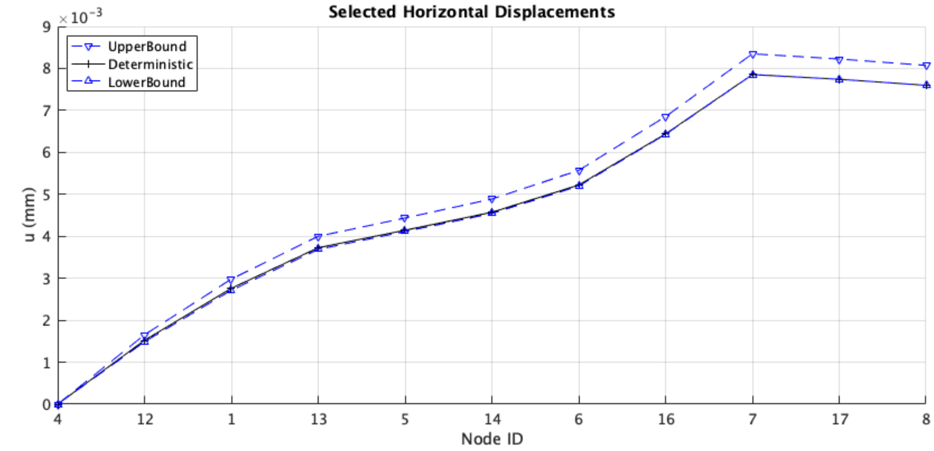



*Figure 14.* The rectangular plate with inhomogeneous material - deformed shape.



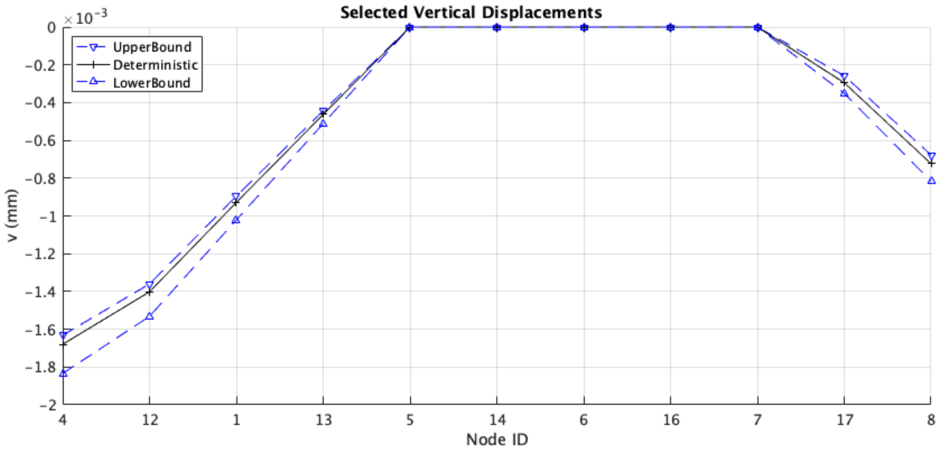




*Figure 15.* The horizontal and vertical displacements of the selected nodes.

## 4. Conclusion

This work presents an INN-based constitutive material model and integrates it into mechanical analysis procedure. In general, obtaining a mathematical constitutive material model which can exactly describe the mechanical behavior is not feasible for all materials. Moreover, the uncertainty in the experimental measurements is not taken into account in such conventional material models. The proposed approach provides a flexible substitution for constitutive material models that accounts for uncertainty in measurements which can naturally be presented as interval-valued data. In this method, an extremal approach is used to construct an INN which is trained by interval-valued data obtained from standard mechanical experiments. The trained INN describes the mechanical behavior of the material and is employed as a substitution for conventional material models by predicting the required material properties in IFEM analysis.

The numerical examples demonstrate that the employed INN can perfectly capture the mechanical behavior of a material under uncertainty and this approach can be used as a flexible alternative to constitutive material models in analyzing mechanical structures built of inhomogeneous materials where providing a closed-form constitutive model is not feasible and/or the conventional mathematical models cannot accurately reflect the mechanical behaviors. In this paper, although a set of synthetic interval-valued data is generated by simulating a standard tensile test and used to train the INN, real experimental data can be used with no additional consideration.

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