Local Reliability Based Sensitivity Analysis with the Moving Particles Method

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Abstract. Local reliability sensitivity methods aim at computing the partial derivatives of the failure probability or the reliability index with respect to model parameters. For efficient local reliability based sensitivity analysis, it is important to avoid repeated evaluations of the performance function. To this end, an extension of the moving particles method to local reliability based sensitivity analysis is presented that is completely based on the already evaluated samples for the reliability estimate and thus avoids repeated evaluations of the performance function. The method is discussed in detail and illustrated by means of examples.

Keywords: local sensitivity analysis, reliability, multilevel splitting, moving particles

1. Introduction

Reliability based sensitivity analysis investigates the dependence of the failure probability on model parameters. In this study, the relevant model parameters are distribution parameters related to the probabilistic characterization of the model input. For sensitivity analysis, local and global approaches can be distinguished. Local reliability sensitivity methods compute the partial derivatives of the failure probability or the reliability index with respect to model parameters. Global reliability sensitivity analysis aims at determining the influence of a model parameter on the failure probability over the entire range of possible values for this parameter. It is often based on Sobol' indices.

(Wu, 1994) considered local reliability based sensitivity analysis and computed partial derivatives of the failure probability by means of importance sampling without any additional evaluations of the performance function. (Song et al., 2009) extended this approach to subset simulation by computing partial derivatives of the conditional probabilities for each of the subsets.

Here, the reliability analysis is carried out by the moving particles method, cf. (Walter, 2015), where a threshold is associated to each sample, samples are moved to new positions in the design space and the number of moves for the initial samples (the particles) to reach the failure region are counted and yield an estimator for the failure probability, which is based on a Poisson process. The estimator is of comparable accuracy and efficiency as the subset simulation estimator. For local reliability based sensitivity analysis, it is important to avoid repeated evaluations of the performance function. To this end, an extension of the moving particles method to local reliability based sensitivity analysis is presented. It is shown that the method is not only completely based

on the already evaluated samples for the reliability estimate and thus avoids repeated evaluations of the performance function, but that it can be carried out after the last move of a particle and just once after the reliability estimation has been carried out.

The paper is organized in five sections. After the introduction, a brief outline of the moving particles method is given. Following this, local reliability sensitivity analysis is integrated in the moving particles method and it is shown that only a single post processing step is required which is carried out when the last particle has been moved into the failure domain. Finally, the accuracy and efficiency of the proposed method is investigated by examples and conclusions are drawn.

2. Brief Outline of the Moving Particles Method

Structural reliability analysis is concerned with the computation of the failure probability

$$P_F = \int_{g(x)<0} p(x,\theta)\mathrm{d}x, \tag{1}$$

where the real valued function $g(x)$, $x \in \mathbb{R}^n$, denotes the performance function of the structure, $p(x,\theta)$ is the probability density function of those structural parameters that are assumed to be random variables and θ contains the parameters of the probability distributions. The limit state function $g(x) = 0$ separates the safe domain $\{x \in \mathbb{R}^n : g(x) > 0\}$ from the failure domain $F = \{x \in \mathbb{R}^n : g(x) < 0\}$. In engineering applications, the failure probability is small and direct Monte Carlo simulation for its estimation requires a huge number of evaluations of the performance function and thus a large computational effort.

In order to reduce the computational effort, importance sampling and importance splitting have been widely applied to the estimation of rare events (Schuëller et al., 2004). Importance sampling estimates the occurrence of rare events by generating samples from an alternative distribution and correcting for the bias by the introduction of weights. The success of this method relies on the quality of the importance sampling density, which is efficiently constructed in an adaptive way.

Importance splitting allows estimating small failure probabilities efficiently, even for problems that involve a high-dimensional vector of input random variables (Schuëller et al., 2004). The general idea of importance splitting is to discard samples that are too far away from a threshold and to regenerate new samples from the remaining ones. In order to maintain the independence of the samples, Markov Chain Monte Carlo simulation starting from remaining samples is frequently employed.

For time-independent problems, subset simulation (Au and Beck, 2003) is a very efficient method for a wide range of applications. This method considers nested subsets generated by levels of the performance function. These nested subsets yield a multiplicative decomposition of the failure probability in larger conditional probabilities. For the estimation of these conditional probabilities,

conditional samples are generated by means of Markov Chain Monte Carlo simulation.

The moving particles algorithm (Guyader et al., 2011) can be considered as a special case of subset simulation with a maximum number of subsets. It starts with an initial Monte Carlo simulation (MCS) with N samples $x_j^{(1)} \in \mathbb{R}^n$, $j = 1, ..., N$, distributed according to the probability density function $p(x, \theta)$. The values $g(x_j^{(1)})$, $j = 1, ..., N$, of the N samples are ranked. These initial samples (the particles) are then moved to the failure region by the following procedure: For the sample with the maximum value of the performance function (thus furthest away from a limit state), a Markov chain Monte Carlo simulation (MCMC) is carried out starting from one of the other samples and the next state of the Markov chain is accepted, if the value of the performance function can be reduced. The Markov chain can be generated either by application of the classical Metropolis-Hastings algorithm or by direct sampling from a normal transition kernel and requires a burn-in period in order to maintain independence of the samples. This algorithm is repeated and stops if all the particles have been moved into the failure domain. The total number of moves M_T is counted.

Interpreted as a special case of subset simulation with a maximum number of domains, each move of a particle gives rise to a subset $F_i = \{x \in \mathbb{R}^n : g(x) < g_i\}$, $i = 1, ..., M_T$, where $g_i = \max_{1 \leq j \leq N} g(x_j^{(i)})$, $i = 1, ..., M_T$, denotes the maximum value of the performance function of the N particles $x_j^{(i)}$, $j = 1, ..., N$, when carrying out the ith move. For the conditional probability

$$P_i = P(G < g_i | G < g_{i-1}), \tag{2}$$

where for simplicity, the random variable $g(X)$ is abbreviated by G, one finds the estimator

$$\hat{P}_i = \frac{N-1}{N}, \tag{3}$$

because only one particle is moved per each iteration step of the algorithm. $F_0 = \mathbb{R}^n$ together with F_i, $i = 1, ..., M_T$, is a finite sequence of nested subsets and

$$F = \bigcap_{i=0}^{M_T} F_i. \tag{4}$$

Therefore, one finds for the failure probability

$$P_F = \prod_{i=1}^{M_T} P(F_i | F_{i-1}) = \prod_{i=1}^{M_T} P_i \tag{5}$$

and thus

$$\hat{P}_F = \left(\frac{N-1}{N} \right)^{M_T} \tag{6}$$

is an estimator for P_F.

In addition, this algorithm has the advantage that the number of moves of the particles is a Poisson distributed random variable that can be directly related to the failure probability P_F. Thus, properties of the Poisson distribution can be used to analyze the algorithm. For each initial sample j, $j = 1, ..., N$, the number M_j of moves until it reaches the failure region is counted. As has been shown in (Guyader et al., 2011), the number of moves to get an initial sample into the failure region follows a Poisson distribution with parameter $\lambda = -\log P_F$. The estimator for the parameter of the Poisson distribution is obtained from $\lambda = E[M]$ as

$$\hat{\lambda} = \frac{\sum_{j=1}^{N} M_j}{N} = \frac{M_T}{N}. \tag{7}$$

In order to obtain an unbiased estimate, it is mandatory that the trajectories of the Poisson process generated from the initial samples remain independent until the samples finally reach the failure domain. In (Walter, 2015), two means are proposed to maintain the independence:

- Burn-in: The Markov chain simulation is carried out with a burn-in period. The burn-in should ensure the independence of the candidate and the seed of the Markov chain.

- Seed avoidance: Repeated use of the same starting point for the Markov chain should be avoided. Once a sample has been used as starting point, the sample and its offspring should not be used as starting point again.

3. Local Reliability Sensitivity Analysis with the Moving Particles Method

Applying the product rule to equation (5), one finds for the partial derivative of the failure probability with respect to θ:

$$\frac{\partial P_F}{\partial \theta} = P_F \sum_{i=1}^{M_T} \frac{1}{P_i} \frac{\partial P_i}{\partial \theta}. \tag{8}$$

With

$$P(G < g_{i-1}) = \int I_{g<g_{i-1}}(x) p(x) \mathrm{d}x \tag{9}$$

and

$$P_i = P(G < g_i | G < g_{i-1}) = \int I_{g<g_i}(x) \frac{I_{g<g_{i-1}}(x) p(x)}{P(G < g_{i-1})} \mathrm{d}x, \tag{10}$$

where the indicator function is given by

$$I_{g<g_i}(x) = \begin{cases} 1, & \text{if } g(x) < g_i, \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$

the partial derivative of the conditional probability P_i is given by

$$\begin{aligned}\frac{\partial P_i}{\partial \theta} &= \frac{\partial}{\partial \theta}\int I_{g<g_i}(x)\frac{I_{g<g_{i-1}}(x)p(x)}{P(G<g_{i-1})}\mathrm{d}x \\ &= \int I_{g<g_i}(x)\frac{I_{g<g_{i-1}}(x)}{P(G<g_{i-1})}\frac{\partial p(x)}{\partial \theta}\mathrm{d}x - \int I_{g<g_i}(x)\frac{I_{g<g_{i-1}}(x)p(x)}{P(G<g_{i-1})^2}\frac{\partial P(G<g_{i-1})}{\partial \theta}\mathrm{d}x \\ &= \int I_{g<g_i}(x)\frac{1}{P(G<g_{i-1})}\frac{\partial p(x)}{\partial \theta}\mathrm{d}x - \int I_{g<g_i}(x)\frac{p(x)}{P(G<g_{i-1})^2}\frac{\partial P(G<g_{i-1})}{\partial \theta}\mathrm{d}x,\end{aligned} \tag{12}$$

as $g_i < g_{i-1}$ and thus $I_{g<g_i}(x)I_{g<g_{i-1}}(x) = I_{g<g_i}(x)$. The last expression can also be written as

$$\frac{\partial P_i}{\partial \theta} = E_{G<g_{i-1}}[I_{g<g_i}(x)\frac{\partial \ln p(x)}{\partial \theta}] - \frac{P_i}{P(G<g_{i-1})}\frac{\partial P(G<g_{i-1})}{\partial \theta}. \tag{13}$$

The expression in the last term involves the partial derivative of the probability related to a set with larger bound, namely F_{i-1}. Analogously to equation (8), we find the expression

$$\frac{\partial P(G<g_{i-1})}{\partial \theta} = P(G<g_{i-1})\sum_{j=1}^{i-1}\frac{1}{P_j}\frac{\partial P_j}{\partial \theta} \tag{14}$$

for this partial derivative. In summary, the expression for the partial derivative of the conditional probability P_i with respect to θ reads

$$\frac{\partial P_i}{\partial \theta} = E_{G<g_{i-1}}[I_{g<g_i}(x)\frac{\partial \ln p(x)}{\partial \theta}] - \sum_{j=1}^{i-1}\frac{1}{P_j}\frac{\partial P_j}{\partial \theta}. \tag{15}$$

Inserting this expression into equation (8) and observing that $\hat{P}_1 = \hat{P}_2 = ... = \hat{P}_{M_T}$ yields a telescoping sum, such that the estimator for the sensitivity of the failure probability is obtained as

$$\frac{\partial \hat{P}_F}{\partial \theta} = \left(\frac{N-1}{N}\right)^{M_T-1}\left(\frac{1}{N}\sum_{j=1}^{N} I_{g<g_{M_T}}(x_j^{(M_T)})\left.\frac{\partial \ln p(x)}{\partial \theta}\right|_{x=x_j^{(M_T)}}\right). \tag{16}$$

Thus, the score function $\frac{\partial \ln p(x)}{\partial \theta}$, cf. (Rubinstein and Shapiro, 1993), is evaluated and averaged for the remaining $N-1$ particles before the move of the last particle into the failure domain.

For the n-dimensional normal probability density function

$$p(x,\mu,\Sigma) = \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}}\exp(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)) \tag{17}$$

the partial derivatives with respect to μ and Σ yield the expressions

$$\frac{\partial \ln p(x,\mu,\Sigma)}{\partial \theta} = \Sigma^{-1}(x-\mu) \tag{18}$$

Table I. Distribution parameters for the random variables in example 1.

	q [N/m]	ℓ [m]	A_C [m^2]	E_C [N/m^2]	A_S [m^2]	E_S [N/m^2]
mean value	20000	12	0.04	2×10^{10}	9.82×10^{-4}	1×10^{11}
std. dev.	1400	0.12	0.0048	1.2×10^{9}	5.892×10^{-5}	6×10^{9}

for $\theta = \mu$ and

$$\frac{\partial \ln p(x,\mu,\Sigma)}{\partial \theta} = -\Sigma^{-1} + \frac{1}{2}(\Sigma^{-1} \circ I) + \Sigma^{-1}(x-\mu)(x-\mu)^T\Sigma^{-1} - \frac{1}{2}(\Sigma^{-1}(x-\mu)(x-\mu)^T\Sigma^{-1} \circ I) \quad (19)$$

for $\theta = \Sigma$, where $\circ$ denotes the Hadamard product and I the $n \times n$ identity matrix.

4. Examples

Example 1: Roof truss

Consider the truss structure shown in Figure 1, cf. also (Song et al., 2009). The top boom and the compression bars are made of reinforced concrete (cross section area A_C, Young's modulus E_C), while the bottom boom and the tension bars are made of steel (cross section area A_S, Young's modulus E_S). The concentrated load P represents a constant distributed load q that is applied on the roof truss and thus $P = \frac{q\ell}{4}$. The parameters q, ℓ, A_C, E_C, A_S and E_S are assumed to be independent normally distributed random variables. Table I displays the mean values and standard deviation of the random variables which are the same as in (Song et al., 2009).

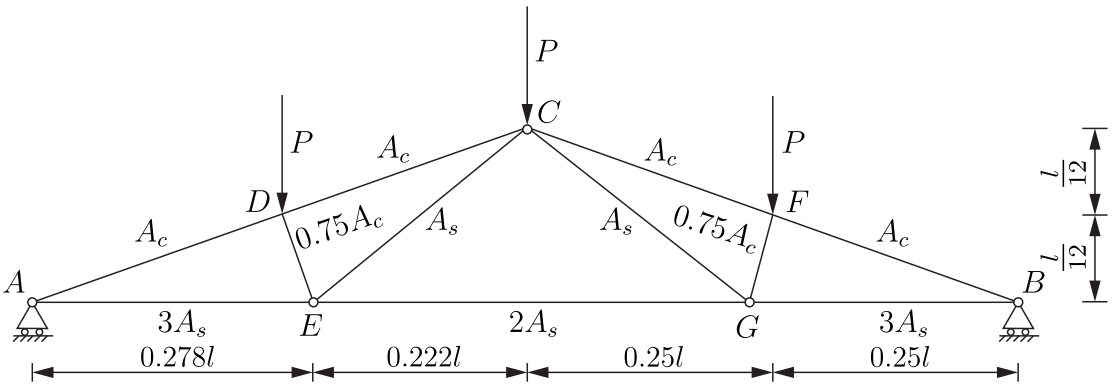

Figure 1. Roof truss structure.

Table II. Example 1: Failure probability and sensitivity of the failure probability with respect to the distribution parameters.

	P_F	$\frac{\partial P_F}{\partial m_q}$	$\frac{\partial P_F}{\partial m_\ell}$	$\frac{\partial P_F}{\partial m_{A_C}}$	$\frac{\partial P_F}{\partial m_{E_C}}$	$\frac{\partial P_F}{\partial m_{A_S}}$	$\frac{\partial P_F}{\partial m_{E_S}}$
estimate	0.00937	1.10e-5	0.0403	-2.110	-3.71e-12	-186	-1.81e-12
elasticity		23.5	51.9	-9.1	-8.0	- 19.6	-19.5
c.o.v.	0.05	0.05	0.08	-0.06	-0.08	-0.07	-0.06
rel. error [%]	0.5	0.7	0.9	1.0	1.2	0.1	0.6
		$\frac{\partial P_F}{\partial \sigma_q}$	$\frac{\partial P_F}{\partial \sigma_\ell}$	$\frac{\partial P_F}{\partial \sigma_{A_C}}$	$\frac{\partial P_F}{\partial \sigma_{E_C}}$	$\frac{\partial P_F}{\partial \sigma_{A_S}}$	$\frac{\partial P_F}{\partial \sigma_{E_S}}$
estimate		1.57e-5	0.0182	2.5047	1.93e-12	204	1.99e-12
elasticity		2.4	0.2	1.3	0.2	1.3	1.3
c.o.v.		0.06	0.22	0.08	0.20	0.09	0.09
rel. error [%]		2.9	1.4	1.3	3.4	0.8	1.0

If the displacement of the top node should not exceed 3 cm, the limit state function reads

$$g(q, \ell, A_C, E_C, A_S, E_S) = 0.03 - \frac{q\ell^2}{2}\left(\frac{3.81}{A_C E_C} + \frac{1.13}{A_S E_S}\right) \tag{20}$$

Table II summarizes the results for the estimated failure probability and the obtained sensitivities with respect to the distribution parameters. In order to be able to compare the influence of the parameters, it also states the elasticity, which is given by

$$\frac{\theta}{P_F}\frac{\partial P_F}{\partial \theta}, \tag{21}$$

cf. (Lemaire, 2010). The initial number of samples, i.e. the particles, has been set to 2000. The relative error has been computed with respect to results obtained by Monte Carlo simulation and published in (Song et al., 2009).

The results indicate that the mean values, notably the mean length ℓ and the mean distributed force q are most influential on the failure probability. The approximation error is rather low. The proposed method based on just 2000 samples is as exact as a Monte Carlo simulation with 10^7 samples. This clearly demonstrates the efficiency of the proposed method. Also, most of the coefficients of variation (c.o.v.) are rather low. However, it was noted that for the sensitivity of the failure probability with respect to the length ℓ and the Young's modulus of concrete E_C, the coefficient of variation is slightly increased. A possible reason could be the fact that the proposed method computes an average on the 2000 samples, which is a rather low number.

Example 2: One storey one bay elastoplastic frame

This example deals with a series system consisting of four failure modes that lead to nearly the same first order reliability index β. Consider the elastoplastic frame structure shown in Figure 2. The four potential failure modes yield the performance functions

$$\begin{aligned} g_1 &= 2M_1 + 2M_3 - 4.5S, \\ g_2 &= 2M_1 + M_2 + M_3 - 4.5S, \\ g_3 &= M_1 + M_2 + 2M_3 - 4.5S, \\ g_4 &= M_1 + 2M_2 + M_3 - 4.5S, \end{aligned} \tag{22}$$

cf. (Song et al., 2009). The performance function is thus given by the minimum of these four functions:

$$g(M_1, M_2, M_3, S) = \min(g_1, g_2, g_3, g_4). \tag{23}$$

M_i, $i = 1, .., 3$ and S are assumed to be independent normally distributed random variables. For M_i, $i = 1, ..., 3$, a mean value of 5.2872 and a standard deviation of 0.1492 is assumed, while for S, the mean value was 3.8378 and the standard deviation 0.3853, so that the reliability index β for each of the failure modes is 3.334 for the first failure mode and 3.364 for the others.

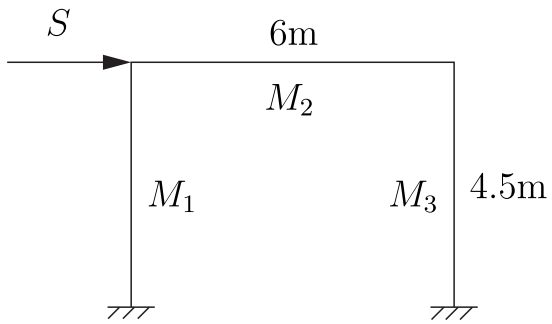

Figure 2. One storey one bay elastoplastic frame.

The results for the estimated failure probability and the obtained sensitivities are collected in Table III. The initial number of samples, i.e. the particles, has been set to 2000. Again, the relative error has been computed with respect to Monte Carlo simulation results with 10^7 samples published in (Song et al., 2009).

As Table III demonstrates, the mean values and the standard deviation related to the applied force S influence the failure probability most. The approximation error is always less than 2%. This confirms the findings of the previous example that the proposed method is very efficient and as exact as a highly accurate Monte Carlo simulation with 10^7 samples. As in the previous example, some of the coefficients of variation are slightly increased (notably for the sensitivities with respect to the standard deviation of the parameters M_i, $i = 1, ..., 3$).

Table III. Example 2: Failure probability and sensitivity of the failure probability with respect to the distribution parameters.

	P_F	$\frac{\partial P_F}{\partial m_{M_1}}$	$\frac{\partial P_F}{\partial m_{M_2}}$	$\frac{\partial P_F}{\partial m_{M_3}}$	$\frac{\partial P_F}{\partial m_S}$
estimate	0.0182	-0.0387	-0.0236	-0.038	0.1132
elasticity		-11.2	-6.9	-11.0	23.9
c.o.v.	0.05	-0.11	-0.18	-0.13	0.04
rel. error [%]	0.4	1.4	1.3	0.5	0.1
		$\frac{\partial P_F}{\partial \sigma_{M_1}}$	$\frac{\partial P_F}{\partial \sigma_{M_2}}$	$\frac{\partial P_F}{\partial \sigma_{M_3}}$	$\frac{\partial P_F}{\partial \sigma_S}$
estimate		0.0177	0.0195	0.0171	0.2318
elasticity		0.1	0.2	0.1	4.9
c.o.v.		0.29	0.3	0.34	0.05
rel. error [%]		1.2	0.6	3.0	0.1

Example 3: High-dimensional problem

This example deals with a component reliability problem of dimension 100. The performance function is given by

$$g(X) = \frac{1}{1000 + \sum_{i=1}^{100} X_i} - \frac{1}{1000 + 3\sqrt{100}}, \tag{24}$$

where the 100 random variables are normal with mean $m = 0$ and standard deviation $\sigma = 1$, cf. (Rahman, 2009). Two cases are considered: (a) the random variables are independent, (b) all random variables are correlated with correlation coefficient $\rho = 0.5$.

As Table IV indicates, the high dimension of the vector of random parameters does not affect the accuracy of the obtained results. Also the correlation of the random variables seems to have no influence on the accuracy. However, it is observed that for the estimator of the partial derivative of the failure probability with respect to the standard deviation, the coefficient of variation is larger than for the uncorrelated case.

5. Conclusions

In this paper, local reliability based sensitivity analysis based on the moving particles method has been introduced. It has been shown that the sensitivity analysis can be carried out in a single post processing step after the last particle has been moved into the failure domain.

Table IV. Example 3: Failure probability and sensitivity of the failure probability with respect to the distribution parameters.

$\rho = 0$	P_F	$\frac{\partial P_F}{\partial m}$	$\frac{\partial P_F}{\sigma}$
estimate	1.35e-3	4.44e-2	1.32e-2
c.o.v.	0.069	0.069	0.083
rel. error [%]	0.3	0.1	1.0
$\rho = 0.5$	P_F	$\frac{\partial P_F}{\partial m}$	$\frac{\partial P_F}{\sigma}$
estimate	0.336	0.514	0.156
c.o.v.	0.024	0.027	0.70
rel. error [%]	0.1	0.1	1.3

The examples demonstrate that the proposed method yields accurate estimates for the partial derivatives of the failure probability with respect to the distribution parameters of the random variables. However, due to the low number of particles on which the estimator is built, the coefficient of variation can be rather high. In this regard, a multilevel approach (Proppe, 2020) would certainly help to reduce the coefficient of variation and thus to further improve the efficiency of the sensitivity analysis.

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