Imprecise subset simulation for reliability analysis

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Abstract. The objective of this work is to quantify the uncertainty in structural reliability that results from lack of available data, necessary to precisely identify probability distributions for the model input parameters. To this end, we propose a computationally efficient framework that incorporates multimodel Bayesian/information theoretic method for probabilistic inference in the context of the popular Subset simulation (SuS), in order to estimate the uncertainty in the probability of failure estimates for a given structure. The proposed method consists of two steps: First, a multi-model Bayesian/information theoretic framework is utilized in order to assess the model-form and parametric uncertainty of the input random variables. Subsequently, imprecise SuS is conducted for the identified family of distributions. The outcome of the proposed method is imprecise probabilities of given response quantities in the form of probabilities of probabilities that allows us to bound response quantities of interest in a probabilistic manner and therefore assess confidence in our probabilistic estimates.

Keywords: Subset simulation, imprecise probabilities, structural reliability

1. Introduction

Various methods have been developed over the last decades to estimate the reliability of an engineering system, such as the widely-used First and Second Order Reliability methods (FORM/SORM) (Rackwitz and Fiessler, 1978; Breitung, 1989), response surface methods (Faravelli, 1989), as well as improved simulation approaches such as the robust Monte Carlo simulation (MCS) (Fishman, 1996; Rubinstein, 1981), variance reduction methods based on Markov Chain Monte Carlo (MCMC) algorithms (e.g. line sampling (Pradlwarter et al., 2007), importance sampling (Schuëller et al., 2004; Koutsourelakis et al., 2004), and subset simulation (Au and Beck, 2001)) to name just a few. However, the success of these reliability methods is conditional on exact knowledge of the probability distributions of the random variables, which allows mapping the problem in the standard normal space.

In reality, exact distributions for the random variables of a structural system are rarely known. This lack of knowledge limits the applicability of even the most robust reliability methods, despite their many advances. The challenge is thus to infer the distribution models from data that describe the system. However, when the available data are scant, it is often impossible to assign a unique probability distribution model for the random variables, which introduces uncertainty (or imprecision) into the estimate. Along those lines, over the last several years significant attention has been placed on the impact of imperfect knowledge about the probability distributions on the

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evaluation of structural reliability by the scientific community (Hurtado et al., 2012; Fetz and Oberguggenberger, 2016; Zhang and Shields, 2018a; Wang et al., 2012).

In this work we address the challenge of quantifying uncertainty in probability of failure estimates using subset simulation (SuS) when data for inferring distributions are scarse. SuS has proven very successful for wide-ranging problems when the probability models (and hence the conditional levels) are uniquely prescribed. However, when the probability models are uncertain and represented by multiple models, the location of each conditional level is uncertain, which induces uncertainty in probability of failure estimates. The paper is organized as follows: Section 2 presents an overview of subset simulation for structural reliability analysis. In Section 3, we discuss the tools necessary for quantifying model-form and parametric uncertainty from sparse data. In Section 4, the proposed imprecise subset simulation is introduced. Last but not least, in Section 5 we utilize the proposed methodology to conduct uncertainty quantification on the probability of failure estimates using imprecise SuS for a stochastic single degree freedom system under dynamic excitation.

2. Review of subset simulation (SuS)

Consider that the behavior of a structural system is reflected in a performance function $g(\mathbf{x})$, where $\mathbf{x} = [X_1, \ldots, X_n] \in \mathbb{R}^n$ is a vector of the n uncertain parameters of the system. Structural reliability is interested in determining the probability of failure of the system given by

$$P_F = \mathbb{P}(g(\mathbf{x}) \leq 0) = \int_F \mathrm{I}_F(\mathbf{x}) f(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{1}$$

where F is the failure region, $f(\mathbf{x})$ is the joint probability density of $\mathbf{x}$, and I_F is the indicator function: $\mathrm{I}_F(\mathbf{x}) = \mathbf{1}$ if $g(\mathbf{x}) \leq 0$ and $\mathrm{I}_F(\mathbf{x}) = 0$ otherwise. For practical issues, it is preferable to transfer the problem to the standard normal space (with the one-to-one mapping $\mathbf{u} = \mathcal{T}(\mathbf{x})$) where calculations are easier. The probability of failure can be then expressed as

$$P_F = \mathbb{P}(G(\mathbf{u}) \leq 0) = \int_F \mathrm{I}_F(\mathbf{u}) \varphi_n(\mathbf{u}) \mathrm{d}\mathbf{u} \tag{2}$$

where $\varphi_n(\mathbf{u}) = \prod_{j=1}^n \phi_j(u_j)$ and $\phi_j(u_j)$ is the standard normal probability density function and $G(\mathbf{u}) = g(\mathcal{T}^{-1}(\mathbf{u}))$ is the performance function in the standard normal space.

Subset simulation (Au and Beck, 2001) is a variance reduction method that has drawn significant attention because of its efficiency to estimate small failure probabilities in high-dimensional spaces. In SuS, variance reduction is achieved through representation of the failure event in the standard normal space, $F = \{\mathbf{u} \in \mathbb{R}^n : G(\mathbf{u}) \leq 0\}$, as a sequence of M partial failure events $F_1, F_2, \cdots, F_M$, such that $F_1 \supset F_2 \supset \cdots \supset F_M$, and $F = \cap_{i=1}^M F_i$. This division into subsets offers the possibility to transform the simulation of rare events into a set of simulations of more frequent events and evaluate the failure probability P_F as a product of probabilities

$$P_F = \mathbb{P}[F] = \mathbb{P}[\cap_{i=1}^m F_i] = \mathbb{P}[F_1] \prod_{i=2}^m \mathbb{P}(F_i | F_{i-1}) = P_1 \prod_{i=2}^m P_i \tag{3}$$

Each intermediate failure event is defined as $F_i = \{G(\mathbf{u}) \leq b_i\}$, where $b_i > 0$ is a threshold value selected in a way that nearly equal partial failure probabilities $P_i,\ i = 2, \cdots, M$ are obtained for each subset. However, it is difficult to specify in advance the limit values b_i according to a prescribed probability P_i. Therefore the limit values have to be determined adaptively within the simulation. Usually, each conditional probability P_i equals a target value typically chosen between 0.1 and 0.3 (Zuev et al., 2012).

Although the probability $P_1 = \mathbb{P}[F_1]$ can be easily computed through Monte Carlo simulation, calculating the subsequent conditional probabilities is not trivial. It requires generation of samples from the conditional distribution $\varphi_n(\mathbf{u}|F_i)$ given that $\mathbf{u}$ lies in F_i. To this end, several MCMC algorithms have been proposed to drawn samples from $\varphi_n(\mathbf{u}|F_i)$. In the original SuS work (Au and Beck, 2001), a component-wise modified Metropolis-Hasting (MMH) algorithm was introduced. Since that time, A large number of new MCMC algorithms have been introduced. These include algorithms based on repeated generations of the candidate state until the first acceptance criterion is satisfied (Santoso et al., 2011), the related delayed rejection method (Miao and Ghosh, 2011; Au et al., 2012; Zuev et al., 2012), methods that translate gradually samples from the prior to samples from the posterior distribution through a combination of importance sampling and MCMC (Ching and Chen, 2007) and methods to adaptively update the proposal density in each conditional level (Papaioannou et al., 2015). In (Shields et al., 2020) the authors proposed to use the affine invariant ensemble sampler with stretch moves proposed by Goodman and Weare (Goodman and Weare, 2010), to perform SuS for problems where distributions or conditional distributions are strongly non-Gaussian, highly dependent, and/or degenerate. The main benefit of this algorithm is that it allows SuS to be conducted for problems of arbitrary complexity in the original parameter space of random variables (i.e. without transformation of variables to uncorrelated standard normal). For this reason, in this work we utilize the affine invariant ensemble sampler-based SuS.

3. Quantifying model-form and parametric uncertainty from small data

In order to perform reliability analysis of a structural system, knowledge of the distribution model $\mathcal{M}_j$ (with probability density function (pdf) $p_j(x;\theta_j)$ and corresponding parameters θ_j) of each random variable X_j is required. However, in order to be able to identify a unique "best" model among a set of candidate models (selected by the user), inference of collected data $\mathbf{d}$ is necessary.

Consider a set of candidate probability models $\{\mathcal{M}_l\}_{l=1}^{N_d}$ that reflects a range of plausible distributions of the random variable X_j. The multi-model inference framework, first presented by (Burnham and Anderson, 2004) assigns a probability π_l to model $\mathcal{M}_l$ which reflects our confidence that this model is the "best" model (among the candidates models) to represent the data. In this setting, the information theoretic multi-model selection approach, introduced in (Zhang and Shields, 2018a) will be used to calculate these probabilities by measuring the Kullback-Leibler (K-L) information loss ((Kullback and Leibler, 1951)). Alternatively, Bayesian multi-model selection can be used as presented in (Zhang and Shields, 2018b). More specifically, the Akaike Information Criterion (AIC) ((Akaike, 1974)), defined as

$$\mathrm{AIC}^{(l)} = -2\log(\mathcal{L}(\theta_l^\star|\mathbf{d},\mathcal{M}_l)) + 2k_l \tag{4}$$

is used to quantify the information loss from representing the data with candidate model $\mathcal{M}_l$. In this equation, $\mathcal{L}(\cdot)$ is the likelihood function for model $\mathcal{M}_l$ evaluated at the maximum likelihood parameters $\theta_l^\star$ given the data $\mathbf{d}$. Moreover, k_l is the dimension of the parameter vector θ_l and the term $2k_l$ a the bias correction factor. Since the AIC is an asymptotic quantity relying on large data sets, an extension of the AIC introduced by (Hurvich and Tsai, 1989), termed AIC_c, is utilized when dealing with small data sets. The modified criterion incorporates a second-order bias correction term as

$$\text{AIC}_c^{(l)} = -2\log(\mathcal{L}(\theta^\star|\mathbf{d},\mathcal{M}_l)) + 2k_l + \frac{(2k_l^2+k_l)}{n-k_l-1} \tag{5}$$

which is a function of is the size n of the data set and the dimension k_l of the parameter vector.

Model probabilities can be derived from the AIC_c, under the assumption that all prior models have equal probability $1/N_d$, as

$$\pi_l = p(\mathcal{M}_l|\mathbf{d}) = \frac{\exp\left(\frac{-\Delta_A^{(l)}}{2}\right)}{\sum_{l=1}^{N_d}\exp\left(\frac{-\Delta_A^{(l)}}{2}\right)} \tag{6}$$

where $\Delta_A^{(l)} = \text{AIC}_c^{(l)} - \min_j(\text{AIC}_c^{(l)})$ is the relative information loss (in this scale the most probable model has $\Delta_A^{(j)} = 0$).

After quantifying model-form uncertainty, we also need to quantify uncertainties associated with the parameters θ_r of model $\mathcal{M}_r$ where $r = 1,\ldots,m$, where m is the number of models from the original model set having non-neglibile model probability. This is done using standard Bayesian parameter estimation where a prior pdf $p(\theta_r;\mathcal{M}_r)$ is assigned that reflects our current belief about the values of the parameters. Appropriate selection of a non-informative prior distribution is very important when datasets are small (Zhang and Shields, 2018b). A popular and simple choice, though not necessarily the best choice, is the uniform prior $p(\theta_r;\mathcal{M}_r) = 1/m$ which favors all parameters equally within a specified range. The posterior pdf is estimated using Bayes' Rule as:

$$p^\star(\theta_r|\mathbf{d},\mathcal{M}_r) = \frac{p(\mathrm{d}|\theta_r,\mathcal{M}_r)p(\theta_r;\mathcal{M}_r)}{p(\mathrm{d};\mathcal{M}_r)} \propto \mathcal{L}(\theta_r|\mathrm{d},\mathcal{M}_r)p(\theta_r;\mathcal{M}_r) \tag{7}$$

where $\mathcal{L}(\theta_r|\mathrm{d};\mathcal{M}_r) = p(\mathrm{d}|\theta_r;\mathcal{M}_r)$ is the likelihood of the parameters θ_r given the observed data d, and $p(\mathrm{d};\mathcal{M}_r)$ is the evidence estimated by

$$p(\mathbf{d};\mathcal{M}_r) = \int p(\mathbf{d}|\theta_r;\mathcal{M}_r)p(\theta_r;\mathcal{M}_r)d\theta_r \tag{8}$$

Since computation of the evidence is often not straightforward, a Markov Chain Monte Carlo (MCMC) approach is applied to draw samples from the posterior $p^\star(\theta_r|\mathrm{d};\mathcal{M}_r)$.

This two-step multi-model selection process provides set of models $\{\mathcal{M}_r\}_{r=1}^m$ with associated model probabilities π_r and the joint posterior pdf of each model's parameters. In order to use these multimodel probabilities, we randomly sample from the model set and model parameters by means of Monte Carlo sampling to establish a finite set of n_c probabilities models to be used for

reliability analysis, as follows. First, we randomly pick a probability model $\mathcal{M}_r$ according to the model probabilities π_r. We then randomly sample the parameters of that model from the joint posterior density obtained from MCMC. For uncertainty propagation it is shown in (Zhang and Shields, 2018a) that n_c can be made arbitrarily large without adding significant computational expense.

4. Imprecise subset simulation (iSuS)

It is widely accepted that subset simulation is efficient and precise when the probability model on the parameter space is uniquely prescribed. However, when the probability model is uncertain and represented by multiple models, there is currently no means by which to incorporate this uncertainty in reliability estimates short of conducting repeated subset simulations, which comes at extremely large computational cost. That is, if conditional probabilities in SuS are held constant, then threshold value b_i (delineating the conditional levels) changes for distribution model $\mathcal{M}_r$ as illustrated in Fig.1). Thus, a new SuS is required for each of the model, which is clearly intractable, especially for real-world structures.

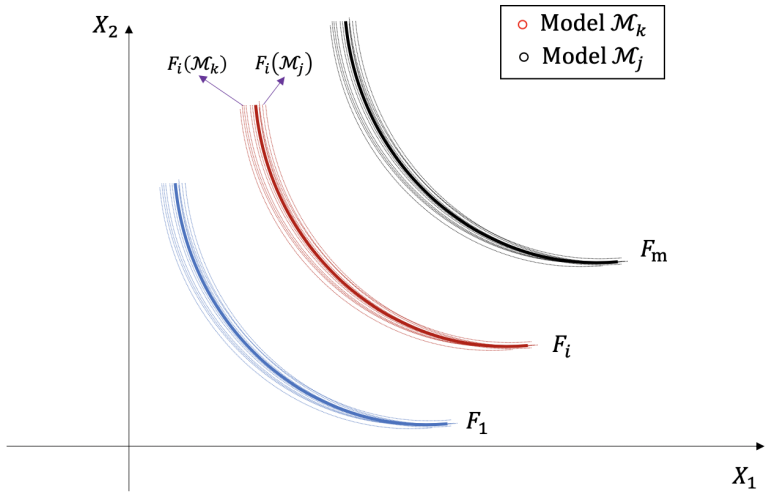

Figure 1. For distribution models $\mathcal{M}_j$ and $\mathcal{M}_k$ running SuS will result in two distinct conditional levels $F_i(\mathcal{M}_j)$ and $F_i(\mathcal{M}_k)$, respectively, if the conditional probability P_i is kept constant.

To overcome this challenge, we propose an approach in which a single SuS is conducted and reliability analysis is conducted using all candidate models are simultaneously. This is achieved by fixing the conditional performance levels F_i, and allowing the conditional probabilities to vary with each probability distribution in the multi-model set, i.e. $P_i \to P_{ir} \equiv P_i(\mathcal{M}_r)$. To this end, an optimal joint sampling density $q_n^\star(\cdot|F_i)$ is identified in order to generate samples that lie in the conditional levels F_i. Then, importance sampling is used to re-weight the samples according to each

distribution in the set and recalculate the conditional probabilities P_{ir} for each model $\mathcal{M}_r$. The probability of failure for model $\mathcal{M}_r$ is then calculated as:

$$P_F(\mathcal{M}_r) = P_1(\mathcal{M}_r)\prod_{i=2}^{m} P_{ir}) \tag{9}$$

A key-point of the proposed approach is to identify the optimal sampling density $q_n^\star(\cdot|F_i)$ from which to sample. This optimal density must be representative of all plausible models and their associated probabilities. (Zhang and Shields, 2018a) derived this optimal sampling density analytical as a probability weighted mixture of the $\{\mathcal{M}_r\}_{r=1}^m$ distribution models. This mixture model is used herein.

5. Steps of the proposed method

The precise steps for the proposed method are: For each variable X_j, $j = 1, \ldots, n$

1. Identify the set of plausible models and their associated model probabilities $\{\mathcal{M}_r, \pi_r\}_{r=1}^m$ using information theoretic multi-model selection.
2. For each model $\mathcal{M}_r$: i) identify the joint posterior parameter probability density $p(\theta_r|\mathbf{d}; \mathcal{M}_r)$ using Bayesian inference and, ii) sample ($n_\theta > 10,000$) from this density using MCMC.
3. Select n_c models from the pool of $m \times n_\theta$ candidate models.
4. Construct the optimal marginal sampling density

$$\hat{q}_j^\star(\mathbf{x}) = \sum_{t=1}^{n_c} \pi_t \mathbb{E}_\theta[p_t(\mathbf{x}|\theta_t)] \tag{10}$$

where $\mathbb{E}_\theta[p_t(\vec{x}|\theta_t]$ is the expectation of all distributions of type $\mathcal{M}_t$ with respect to it's model parameters θ, evaluated by simple Monte Carlo.

The next step is to run a single (optimal) SuS with the following target distribution

$$q_n^\star(\mathbf{x}) = \prod_{j=1}^{n} \hat{q}_j^\star(\mathbf{x}) \tag{11}$$

in order to identify the conditional performance levels F_i^{opt} and obtain samples $\mathbf{x}_i^{opt}$ in each conditional level. Then, we use importance sampling (IS) to re-weight the conditional probabilities P_i for each distribution model $\mathcal{M}_t$. More specifically, at subset i, for each model we calculate the importance weight w_t at each sample point $\mathbf{x}_i^{opt}$ as

$$w_t(\mathbf{x}_i^{opt}) = \frac{p_t(\mathbf{x}_i^{opt})}{q_n^\star(\mathbf{x}_i^{opt})}, \quad t = 1, \ldots, n_c \tag{12}$$

where $p_t(\cdot) = p(\theta_t|\mathbf{d}, \mathcal{M}_t)$. As a result, the conditional probability of failure P_i corresponding to model $\mathcal{M}_t$ is estimated as

$$P_i^t = \frac{1}{N} \sum_{l=1}^{N_c} w_t(\mathbf{x}_l) \mathrm{I}_{F_i^{opt}}(\mathbf{x}_l) \tag{13}$$

Calculate the probability of failure for each model $\mathcal{M}_j$ according to

$$P_F^t = \prod_{i=1}^{m} P_i^t \tag{14}$$

The result is a stochastic set of failure probabilities $\{P_F^t\}_{t=1}^{n_c}$ that incorporate the effects of model form and parametric uncertainty on the true probability of failure.

The SuS with the affine invariant MCMC sampler and the multi-model selection framework used in this work are available in the open-source UQpy Python toolbox (Olivier et al., 2020).

6. Numerical example

A single degree of freedom system having stiffness k and mass m is excited by a sinusoidal load $x(t) = \sin(\omega t)$ with frequency ω rad/sec (see Fig.2). Both the stiffness and mass are consider to be stochastic, normally distributed according to $N(\mu_k, \sigma_k)$ and $N(\mu_m, \sigma_m)$, respectively. The system's equation of motion given by:

$$m\ddot{x} + kx = x(t) \tag{15}$$

In this system, resonance occurs when the natural frequency of the system $\omega_n = \sqrt{\frac{k}{m}}$ becomes equal to the excitation frequency ω of the periodically applied force $x(t)$. To avoid resonance, we consider failure of the system to be associated with the natural frequency being within a threshold ϵ of the excitation frequency ω. That is, failure of the system occurs when $\omega - \epsilon \leq \omega_n \leq \omega + \epsilon$. Hence, the probability of failure is defined as:

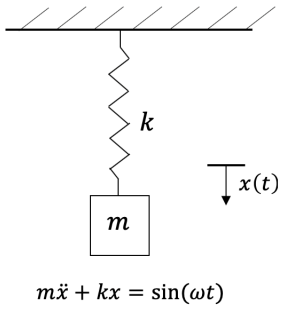

Figure 2. A single degree stochastic system under sinusoidal excitation.

$$P_F = P\left(\omega - \epsilon \leq \omega_n \leq \omega + \epsilon\right) \tag{16}$$

$$= P\left(k - (\omega - \epsilon)^2 m \leq 0 \cup (\omega + \epsilon)^2 m - k \leq 0\right) \tag{17}$$

For the needs of this study we selected $\mu_k = 125$, $\sigma_k = 5$, $\mu_m = 5$, $\sigma_m = 1$, $\omega = 6$ and $\epsilon = 0.0001$. Reliability analysis of the model with means of brute-force Monte Carlo simulation with 10^6 samples gives a probability of failure $P_F = 3.2 \times 10^{-5}$. We also performed 20 independent runs of SuS for the and we calculated a probability of failure $P_F = 3.0 \times 10^{-5}$ with coefficient of variation $c.o.v = 0.40$. Figure 3 shows the samples drawn using a single SuS with 1000 samples per subset level with conditional probability equal to 0.1.

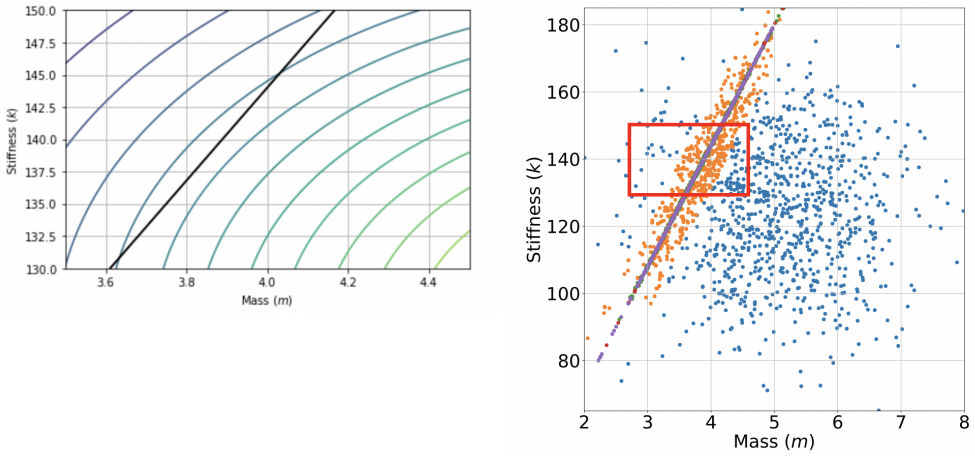

Figure 3. Limit-state function and samples drawn with SuS.

Given limited data from which to infer the distributions of k and m, the multi-model selection framework is applied to each random variable in order to select the "best" probability models from the following distribution families: *lognormal, logistic, inverse Gaussian, Maxwell, Levy* and *normal*. We consider data sets of size 20, 50, 100, 1000 and 10000 in order to showcase the sensitivity of the methods to the number of available data. Figure 4 depicts the probability π of each candidate model as a function of data set size. From this figure we can observe that at least 100 data points are required in order to uniquely identify the true normal distribution for the stiffness and more than 1000 data points are required for the mass.

Next, the joint parameter distribution for each one of the selected models is determined through Bayesian inference. However, for brevity these distributions are not shown. Figure 5 depicts 1000 sampled candidate distributions for each random variable and for increasing data set size. Notice that the clouds of possible distributions narrow with more available data.

To assess the influence of model-form and parametric uncertainty on P_F estimates, we apply the proposed iSuS method. First, we identify the joint optimal sampling density function $q_n^\star(\mathbf{x})$, shown in Figure 6, for different data set sizes. As can be seen from this figure, for increasing number of available data the probability density is converging to the true joint normal distribution.

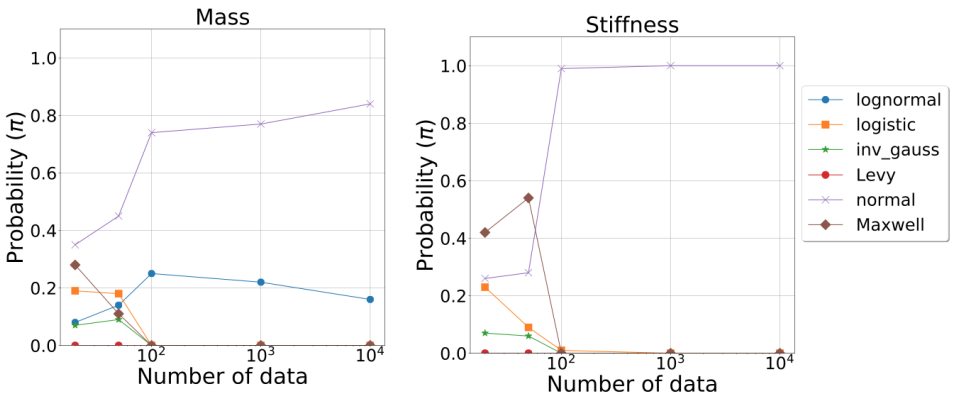

Figure 4. Model probabilities as a function of data set size.

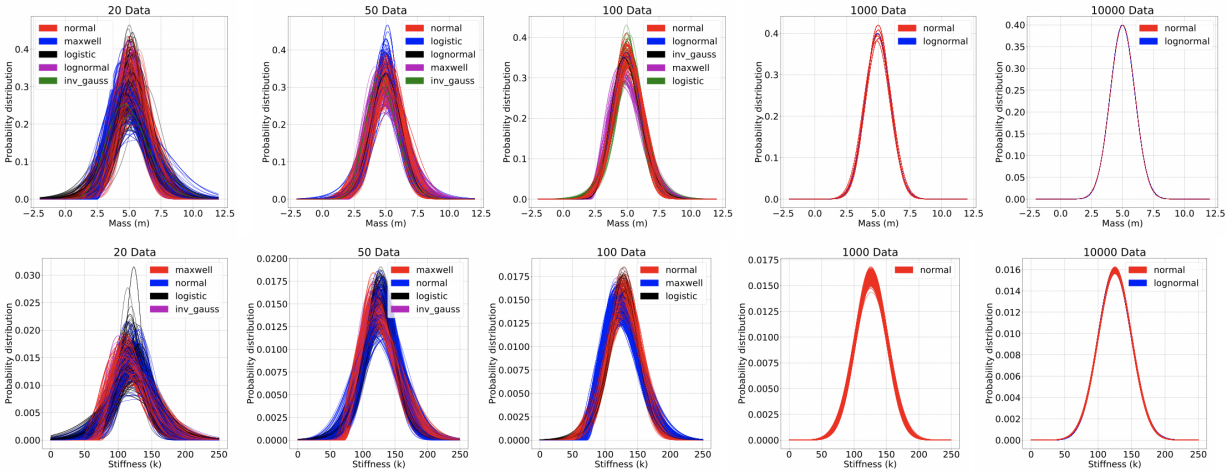

Figure 5. Example 1: Sample sets of probability distributions for different data set sizes for mass m (top) and stiffness k (bottom).

After identifying the optimal sampling distribution we run SuS once, in order to identify the optimal conditional performance levels F_i^{opt} and the conditional samples. Then, the importance weight of each sample is used to weight the conditional probabilities. As a result, each conditional probability of failure is re-evaluated and the probability of failure for the specific model is calculated. Figure 7 shows the resulting mean empirical cumulative distribution functions (from 20 repeated iSuS evaluations) for the probability of failure for the different data set sizes. From this figure we can identify a trend that results in large uncertainty in P_F estimats for small data sets and convergence towards the true P_F for increasing data set size.

We highlight at this point that, with the proposed method assessing uncertainty in P_F estimates requires a single iSuS evaluation, which is equivalent to a single SuS evaluation plus the nominal cost

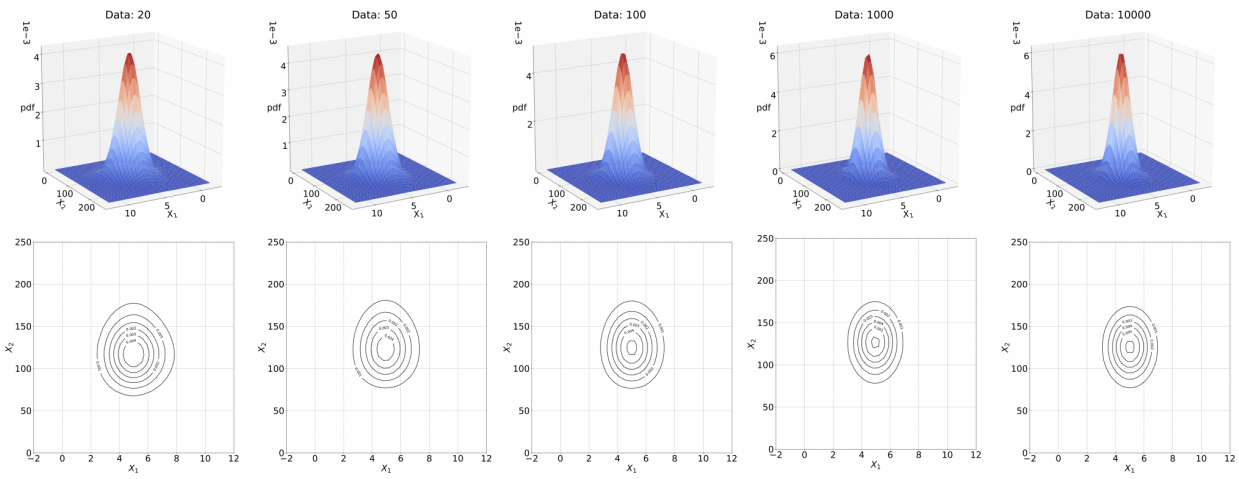

Figure 6. Example 1: Joint probability density $q_n^\star(\mathbf{x})$ (top) with the respective contour (bottom) that is used as the target density in the framework of SuS, for different size of the data set.

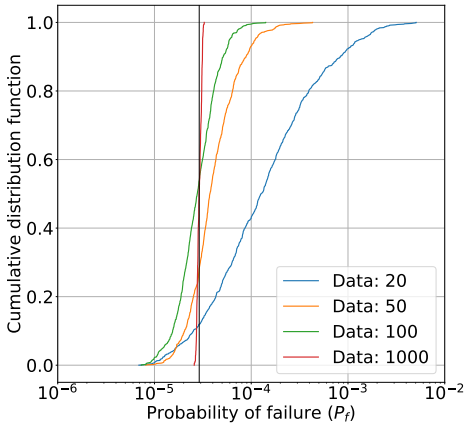

Figure 7. Emperical cdfs of the probability of failure P_F for different data set sizes resulting from 20 iSuS runs. The true $P_F = 2.9 \times 10^{-5}$ is indicated by a vertical black line.

of model reweighting. Accessing the same information using a brute force Monte Carlo approach, requires a new SuS for each model $\mathcal{M}_j$, or a total of 1000 SuS analyses in this case.

7. Conclusions

We propose a method to quantify the uncertainty in structural reliability estimates using subset simulation that results from lack of available data. Our motivation stems from the fact that subset

simulation is very efficient and precise when the probability model on the parameter space (and hence the conditional levels) is uniquely prescribed. However, when the probability model is uncertain and represented by multiple models the location of each conditional level is uncertain which induces uncertainty in the probability of failure estimate. We propose a framework build upon multimodel Bayesian/information theoretic methods and Subset simulation. The outcomes of the proposed method are imprecise probabilities of given response quantities in the form of probabilities of failure probabilities that allows us to bound the probability of failure in a probabilistic manner and therefore assess confidence in our estimates.

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