Elucidating appealing features of differentiable autocorrelation functions: a study on the modified exponential kernel

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Abstract. Research on stochastic processes in recent decades has pointed out that, in the context of modelling spatial or temporal uncertainties, auto-correlation functions that are differentiable at the origin have advantages over functions that are not differentiable. For instance, the non-differentiability of e.g., single exponential autocorrelation functions yields non-smooth sample paths. Such sample paths might not be physically possible or might yield issues when used as random parameters in partial differential equations (such as encountered in e.g., mechanical equilibrium problems). Further, it is known that due to the non-differentiability of certain auto-correlation functions, more terms are required in the stochastic series expansion representations. This makes these representations less efficient from a computational standpoint.

This paper elucidates some additional appealing features of autocorrelation functions which are differentiable at the origin. Further, it focusses on deepening the argumentations in favor of these functions already available in literature. Specifically, focus is placed on single exponential, modified exponential and squared exponential autocorrelation functions. To start, it is shown that the power spectrum of differentiable kernels converges faster to zero wiht increasing frequency as compared to non-differentiable ones. This property allows capturing the same amount of energy of the spectrum with a smaller cut-off frequency, and hence, less stochastic terms. Further, this point is examined with reqards to the Karhunen-Loève series expansion and first and second order Markov processes, generated via auto-regressive representations.

Keywords: stochastic process, exponential kernel, auto-correlation function

1. Introduction

Many of the loads on engineering components, structures and systems, as well as the constitutive properties of these assets exhibit a stochastic nature. This notion is based on the observation that these quantities exhibit apparent variability in time and/or space. In this context, the theoretical framework of stochastic process, and by extension, random fields (Vanmarcke, 1983) has proven

to be an excellent means for capturing inherent (aleatory) uncertainty (Shinozuka and Deodatis, 1991). Stochastic processes represent in essence jointly distributed random variables whose correlation function depends on time and/or space. Throughout literature, efficient methods have been introduced to effectively and accurately sample from these potentially high-dimensional joint distributions.

Typically, the auto-correlation of such stochastic process is governed by a pre-defined auto-correlation function (also often referred to as 'kernel'). This function describes the correlation between two random variables in the stochastic process as a function of the distance in time/space between them. Alternatively, in case the process is stationary, the autocorrelation is governed by the relative distance between two points. The generation of samples from these stochastic processes is usually performed using the well-known Karhunen-Loève series expansion (Betz et al., 2014) or the spectral representation method as introduced by (Rice, 1954), (Borgman, 1969) and later by (Shinozuka and Deodatis, 1991). Also extended methods on these techniques have been introduced. Examples of such methods include the Stochastic Harmonic Function representation by (Chen et al., 2013).

The single exponential autocorrelation kernel has been used extensively in engineering applications. For instance, (Desplentere, Verpoest and Lomov, 2009) applied this kernel to represent a stochastic process to model the permeability in resin transfer molding simulation. Further, in (Cao and Wang, 2014), Bayesian updating in a geo-technical context was done based on a single exponential kernel. The single exponential kernel is often selected due to the availability of analytical solutions to the eigenvalue problem corresponding to the Karhunen-Loève series expansion. However, the appropriate selection of the auto-correlation function is of large importance for the correct modelling and simulation of the phenomenon or property under consideration.

More relevantly to the problem discussed herein, (Ching and Phoon, 2018) studied the effect of the auto-correlation function on the probability of failure in several geo-technical examples. In this work, the authors showed that the smoothness of a sample path had a significant effect on the probability of failure. This is particularly true when no spatial averaging is present in the considered problem capable of smoothing out local variations. Further, it was shown that the sample path smoothness depends on the functional form of the auto-correlation function, and, more precisely, on the differentiability of such function at zero-lag. Thus, this work clearly illustrated that the auto-correlation function ends up affecting the probability of failure.

Furthermore in this context, (Wang et al., 2019) noticed that the application of the single exponential auto-correlation kernel tends to underestimate the failure risk in the unsaturated slope risk assessment. In addition, they pointed out how the differentiability of the auto-correlation function affects the truncation order of the Karhunen-Loève series expansion. Also in this regard, (Sudret and Der Kiureghian, 2000) showed that square exponential kernels are preferable over single exponential kernels, allowing for a highly exact series expansion representation with far fewer terms in the expansion. Similar observations were made by (Sclavounos, 2012) in the context of modelling stochastic ocean waves. In essence, they showed that fewer stochastic variables are required to describe stochastic processes with a narrow spectral bandwidth (i.e., spectra that converge relatively fast to zero when the frequency increases).

As a final note to this overview of the wide-spread application of single exponential kernels (versus less frequently used kernels), examination of the literature showed that non-differentiable

auto-correlation functions are used to represent a random 'stiffness' term in partial differential equations. From a mathematical standpoint, this is not correct since the time derivative of the corresponding sample paths is undefined. To address this, in (Spanos, Beer and Red-Horse, 2007) the "modified exponential kernel" was introduced to alleviate the shortcomings of the traditional single exponential kernel caused by non-differentiability. Further, this kernel maintains the enticing properties of the single exponential kernel, such as the availability of analytical solutions and the possibility to characterize quickly varying spatial or temporal phenomena. Also, this kernel is more efficient in terms of the required number of stochastic quantities as compared to the single exponential kernel. These properties are obtained by solving the issue of non-differentiability at zero-lag in the single exponential kernel, while maintaining its functional form to a great extent.

This paper builds upon the work that was presented in (Spanos, Beer and Red-Horse, 2007) in three ways. First, it provides additional explanations of why this particular kernel is highly effective by analytically deriving the energy error rate convergence in the frequency domain. Second, it compares the modified exponential kernel to another widely used autocorrelation kernel, namely the squared exponential kernel. Finally, it provides additional numerical evidence for the effectiveness of the modified exponential kernel in conjunction with the Karhunen-Loève series expansion, and auto-regressive model representations. The paper is structured as follows. Section 2 starts by introducing some basic concepts related to the modelling and simulation of stochastic processes. Section 3 discusses the convergence of the considered autocorrelation kernels both in the frequency and time domain and discusses the implications hereof for numerical analysis purposes. Finally, Section 4 briefly comprises the conclusions that can be drawn from this work.

2. Spectral stochastic process representation

2.1. Definition and stochastic properties

A finite-dimensional stochastic process $f(\mathbf{t},\theta)$ describes a set of correlated random variables $f(\theta)$ that are assigned to a countable number of locations $\mathbf{t} \in \Omega$ in the model domain $\Omega \subset \mathbb{R}^d$ with dimension $d \in \mathbb{N}$. Each such a random variable $f(\theta) : (\Theta, \varsigma, P) \mapsto \mathbb{R}$, with $\theta \in \Theta$ a coordinate in sample space Θ and ς the sigma-algebra, as such maps from a complete probability space to the real domain. This map holds as long as $f(\mathbf{t},\theta) \in \mathcal{L}^2(\Theta, P)$, with $L^2(\Theta, P)$ the Hilbert space of second-order random variables (i.e., finite variance). For a given event $\theta_i \in \Theta$, $f(t,\theta_i)$ is a realization of the stochastic process. A stochastic process is considered Gaussian if the distribution of $(f(\mathbf{t}_1,\theta), f(\mathbf{t}_2,\theta), \ldots, f(\mathbf{t}_n,\theta))$ is jointly Gaussian $\forall \mathbf{t} \in \Omega$.

Consider $f(t,\theta)$ to be a zero-mean one-dimensional univariate stochastic process (i.e, $\Omega \subset \mathbb{R}$) with constant variance over the domain and autocorrelation function $R_{ff}(\tau) : \Omega \times \Omega \mapsto [0,1]$ and $\tau \subset \Omega$ a lag parameter. The autocorrelation $R_{ff}(\tau)$ of such a stochastic process represents the correlation between two random variables $f(t,\theta)$ and $f(t+\tau,\theta)$, separated by a lag τ:

$$R_{ff}(\tau) = \frac{CV\left[f(t_i,\theta) f(t+\tau,\theta)\right]}{\sqrt{V\left[f(t,\theta)\right]}\sqrt{V\left[f(t+\tau,\theta)\right]}}, \tag{1}$$

with $CV[\cdot,\cdot]$ denoting an operator that returns the covariance and $V[\cdot]$ an operator returning the variance of the argument. In this regard, τ may represent a distance in time or space. In practical

applications, often analytical models for the autocorrelation are applied (Ching and Phoon, 2018), including single exponential $R^s_{ff}(\tau)$, squared exponential $R^{sq}_{ff}(\tau)$ and modified exponential $R^m_{ff}(\tau)$ autocorrelation functions. When considering a single exponential auto-correlation function, the correlation $R^s_{ff}(\tau)$ is expressed as

$$R^s_{ff}(\tau) = \sigma^2 \exp(-|\tau|/b). \tag{2}$$

Similarly, the auto-correlation for a modified exponential $R^m_{ff}(\tau)$ (Spanos, Beer and Red-Horse, 2007) and squared exponential $R^{sq}_{ff}(\tau)$ function are given as

$$R^m_{ff}(\tau) = \sigma^2 \exp(-|\tau|/b)(1 + |\tau|/b) \tag{3}$$

and

$$R^{sq}_{ff}(\tau) = \sigma^2 \exp(-\tau^2/b^2). \tag{4}$$

In these equations b is the so-called correlation length.

In the available literature, single and squared exponential functions are normally used in different application domains. As discussed in (Ching and Phoon, 2018), the main difference between these two autocorrelation functions is the smoothness of the resulting sample paths; related applications to soil engineering can be found in references such as (Yue et al., 2018) and (Yue et al., 2020). The main advantage of a single exponential kernel is the availability of analytical solutions in terms of its Karhunen–Loève expansion (Spanos, 1989) and the capability to characterize quickly varying phenomena. However, it exhibits non-differentiability at zero-lag. The modified exponential autocorrelation function aims at combining the strengths of both the aforementioned kernels, as it provides the temporal/geometric characteristics of the sample paths of a single exponential kernel, it solves the problem of the zero-lag discontinuities, and it provides analytical solutions for its expansion (Spanos, Beer and Red-Horse, 2007).

Alternatively, the auto-correlation of $(f(t_1, \theta), f(t_2, \theta), \ldots, f(t_n, \theta))$ can be represented in the frequency domain by means of a two-sided power spectrum $S_{ff}(\omega) : \Gamma \times \Gamma \mapsto \mathbb{R}$, with $\Gamma \subset \mathbb{R}$ the frequency domain. The Wiener-Khintchine theorem allows for the calculation of the autocorrelation function $R_{ff}(\tau)$ of a stochastic process from its two-sided power spectrum $S_{ff}(\omega)$ and vice versa based on following Fourier transforms:

$$S_{ff}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{ff}(\tau) e^{-i\omega\tau} d\tau \tag{5}$$

and

$$R_{ff}(\tau) = \int_{-\infty}^{+\infty} S_{ff}(\omega) e^{i\omega\tau} d\omega \tag{6}$$

The power spectra corresponding to the autocorrelation functions in Eqs. 2–4 can be shown to b:

$$S^s_{ff}(\omega) = \frac{\sigma^2}{\pi} \frac{b}{b^2\omega^2 + 1}, \tag{7}$$

$$S^m_{ff}(\omega) = \frac{\sigma^2}{\pi} \frac{2b}{(b^2\omega^2 + 1)^2}, \tag{8}$$

and

$$S_{ff}^{sq}(\omega) = \frac{\sigma^2 b}{2\sqrt{\pi}} \exp \frac{-b^2\omega^2}{4}. \tag{9}$$

From Eq. 7 it can be deduced that the energy content of the spectrum is spread out over a higher range of frequencies when considering the case of single exponential kernel. Similarly, it can be noted that the energy content of the modified exponential kernel decreases quadratically with respect to the term $b^2\omega^2 + 1$ in Eq. 8. As such, the bandwidth of the single exponential kernel is wider as compared to the modified exponential. Finally, the $\exp(\cdot)$ term in Eq. 9 suggests that most of the energy content is located in the lower frequencies for the squared exponential kernel. Thus, it is shown that the power-spectra of the zero-lag differentiable kernels, i.e., the modified and squared exponential kernel, converge faster to zero as $\omega \to \infty$.

2.2. Simulation of stochastic processes

2.2.1. *Spectral representation*

Following the power spectral representation of the autocorrelation of a the stochastic process, the simulation of paths of this process can be performed according to (Rice, 1954; Shinozuka and Deodatis, 1991) by

$$f(t,\theta) = \sqrt{2} \sum_{n=0}^{N-1} A_n \cos(\omega_n t + \psi_n(\theta)), \tag{10}$$

where

$$A_n = \sqrt{2 S_{ff}(\omega_n)\Delta\omega}, \tag{11}$$

and ω_n is defined as

$$\omega_n = n\Delta\omega, \tag{12}$$

where $n = 1, \ldots, N-1$ and $\Delta\omega = \omega_u/N$. The phase angles $\psi_n(\theta)$ are considered random and distributed as $\mathcal{U}(0, 2\pi)$, with $\mathcal{U}$ the uniform distribution. Thus, samples $f(t, \theta_i)$ of the stochastic process can be generated by sampling from $\mathcal{U}(0, 2\pi)$. The parameter ω_u represents the cut-off frequency, beyond which the power spectral density function $S_{ff}(\omega)$ may be assumed to be zero for either mathematical or physical reasons. In practice, an energy criterion is commonly used. That is,

$$\int_0^{\omega_u} S_{ff}(\omega_n) d\omega = (1 - e_S) \int_0^{\infty} S_{ff}(\omega_n) d\omega, \tag{13}$$

with e_S a measure for the error, which is typically a small value, e.g., $e = 0.01$ or $e = 0.001$. In this regard, the appropriate selection of ω_u is of paramount importance for the accuracy of the analysis. In essence, it represents the degree of approximation of the energy content used in the representation of the stochastic process. As such, if ω_u is selected too small, a significant portion of the energy of the modelled signal is lost. This might lead to an underestimation of the magnitude of the physical quantity under consideration, which can lead to severe consequences, e.g., underestimation of probability of failure. Further, ω_u cannot be selected arbitrarily larger for numerical reasons. Extremely high values of ω_u require a corresponding quite high number of random variables $\phi_n(\theta)$,

leading to computationally untractable high-dimensional problems. Furthermore note that, while herein attention is focussed on the spectral representation, these arguments are also pertinent to other simulation methods based in the frequency domain, e.g., the Stochastic Harmonic Function representation (Chen et al., 2013).

2.2.2. *Karhunen-Loève expansion*

The Karhunen-Loève expansion is a powerful tool to represent stochastic process (Spanos, 1989). Specifically, following the Karhunen-Loève (KL) series expansion, a stochastic process $f(t,\theta)$ is represented as:

$$f(t,\theta) = \mu_x(t) + \sigma_f \sum_{i=1}^{\infty} \sqrt{\lambda_i}\psi_i(t)\xi_i(\theta), \tag{14}$$

where σ_f is the standard deviation of the random field. The quantities $\lambda_i \in (0,\infty)$ and $\psi_i(t) : \Omega \mapsto \mathbb{R}$ are respectively the eigenvalues and eigenfunctions of the continuous, bounded, symmetric and positive (semi-) definite auto-correlation function $R_{ff}(\tau)$. The decomposition of $R_{ff}(\tau)$ is performed in accordance with Mercer's theorem. That is,

$$R_{ff}(\tau) = \sum_{i=1}^{\infty} \lambda_i \psi_i(t)\psi_i(t'). \tag{15}$$

These quantities are in practice obtained by solving the homogeneous Fredholm integral equation of the second kind. That is,

$$\int_{\Omega} R_{ff}(\tau)\psi_i(t')dt' = \lambda_i\psi_i(t), \tag{16}$$

with $t' = t + \tau$. Analytical solutions to this equation exist only for a limited number of auto-correlation functions. In general, discretization schemes are used to solve this equation, as explained in (Betz et al., 2014). Since $R_{ff}(\tau)$ is bounded, symmetric, positive semi-definite, and, in most practical cases, can be even assumed positive definite, the eigenvalues λ_i are non-negative and the eigenfunctions $\psi_i(t)$ satisfy the following orthogonality condition:

$$\langle \psi_i(t), \psi_j(t) \rangle = \int_{\Omega} \psi_i(t)\psi_j(t)dt = \delta_{ij} \tag{17}$$

with δ_{ij} the Kronecker delta. $\langle \cdot, \cdot \rangle : \Omega \times \Omega \mapsto \mathbb{R}$ denotes the inner product in the functional space. Hence, the eigenfunctions form a complete orthogonal basis on an $\mathcal{L}_2$ Hilbert space. In this case, the series expansion in Eq. 15 can be shown to be optimally convergent (Spanos, 1989).

For practical reasons, the infinite series expansion in Eq. 14 has to be truncated after a finite number of terms $n_{KL} \in \mathbb{N}$. That is,

$$f(t,\theta) = \sigma_f \sum_{i=1}^{n_{KL}} \sqrt{\lambda_i}\psi_i(t)\xi_i(\theta), \tag{18}$$

where n_{KL} should be selected such that following inequality holds:

$$1 - \frac{1}{|\Omega|}\frac{1}{\sigma_f^2}\sum_{i=1}^{n_{KL}} \lambda_i \leq e_\sigma, \tag{19}$$

with e_σ the so-called mean error variance (Betz et al., 2014) and $|\Omega|$ denoting the length of the simulation domain.

In contrast to e_S introduced in Eq. (13), the quantity e_σ does not represent some sort of energy loss measure. Instead, e_σ represents the percentage of the variance of the original process that is captured by the truncated Karhunen-Loève expansion. In this respect, if n_{KL} is selected too small, it will lead to a loss of the variance in the representation. Hence, the magnitude of the physical quantity under consideration will be underestimated. Finally, similarly to the spectral representation, setting n_{KL} too large will render the computational cost of the corresponding analysis untractable due to the high dimension of the parameter input space.

2.3. Auto-regressive models

A third commonly used approach to simulate stochastic processes and fields is the auto-regressive representation (AR) method (Spanos & Zeldin, 1996). According to the AR framework, the value of a stochastic process $f(t,\theta)$ at time t_k with order m can be computed as

$$f(t_k,\theta) = -\sum_{i=1}^{m} a_i f(t_{k-I}) + b_0 w(t_k,\theta), \tag{20}$$

where a_i are the AR parameters and b_0 is the gain factor of the AR model. $w(t_k)$ is a band limited $[-\omega_b,\omega_b]$ white-noise process that satisfies

$$E\left[w(t_k)w(t_l)^T\right] = 2\omega_b I_n \delta_{kl}, \tag{21}$$

where $E[\bullet]$ and $\bullet^T$ denotes the operators of mathematical expectation and transpose respectively, ω_b is the cut-off frequency, I_n is the identity matrix and δ_{kl} the Kronecker delta. The representation in in Eq. (20) is the best linear estimator of $f(t_k,\theta)$ by using the m previous values $[f(t_k), f(t_{k-1}),\ldots,f(t_{k-m})]$. The corresponding error ϵ can be expressed as:

$$\epsilon = \frac{\triangle t}{2\pi} E\left[\left(f(t_k,\theta) + \sum_{i=1}^{m} a_i f(t_{k-I})\right)\right] = b_0^2. \tag{22}$$

The parameters a_i in the series expansion defined in Eq. (20) can generally be determined by minimizing ϵ. This leads to the so-called Yule-Walker equations that relate the stationary target autocorrelation function $R_{ff}(\tau) \equiv R_{ff}(t_k - t_i)$ to the AR parameters via a Toeplitz system of equations:

$$\sum_{i=1}^{m} R_{ff}(t_k - t_i)a_i = -R_{ff}(t_k) \qquad k = 1,\ldots,m \tag{23}$$

Alternative approaches, used to fit an AR model to a predefined autocorrelation model, interpret Eq. (20) as the response of a discrete linear system to a white-noise excitation. For a thorough explanation of the fitting of an AR model to a predefined auto-correlation function or power spectral density, the reader is referred to the seminal work of (Spanos & Zeldin, 1996) or (Spanos & Zeldin, 1998).

3. Convergence of the stochastic process representations

In this section analytical expressions are derived for the energy approximation error of the spectral stochastic representation of the of three considered autocorrelation kernels. Then, the paper provides additional illustrations regarding the improved convergence behavior of the modified exponential kernel over the single exponential when they are used in the context of a KL expansion. Finally, a comparison of AR models for these three autocorrelation functions is included for the sake of completeness.

3.1. Spectral stochastic representation

The convergence behavior of the autocorrelation functions in Eqs. 2-4 is studied with respect to Eq. 13. Herein, analytical expressions for the approximation error e_s are derived with respect to the cut-off frequency ω_u. To determine the cut-off frequency ω_u using Eq. (13), the power spectra $S_{ff}(\omega)$ in Eqs. 7-9 needs to be integrated. The right hand side integral of $S_{ff}(\omega)$ w.r.t. ω is obtained

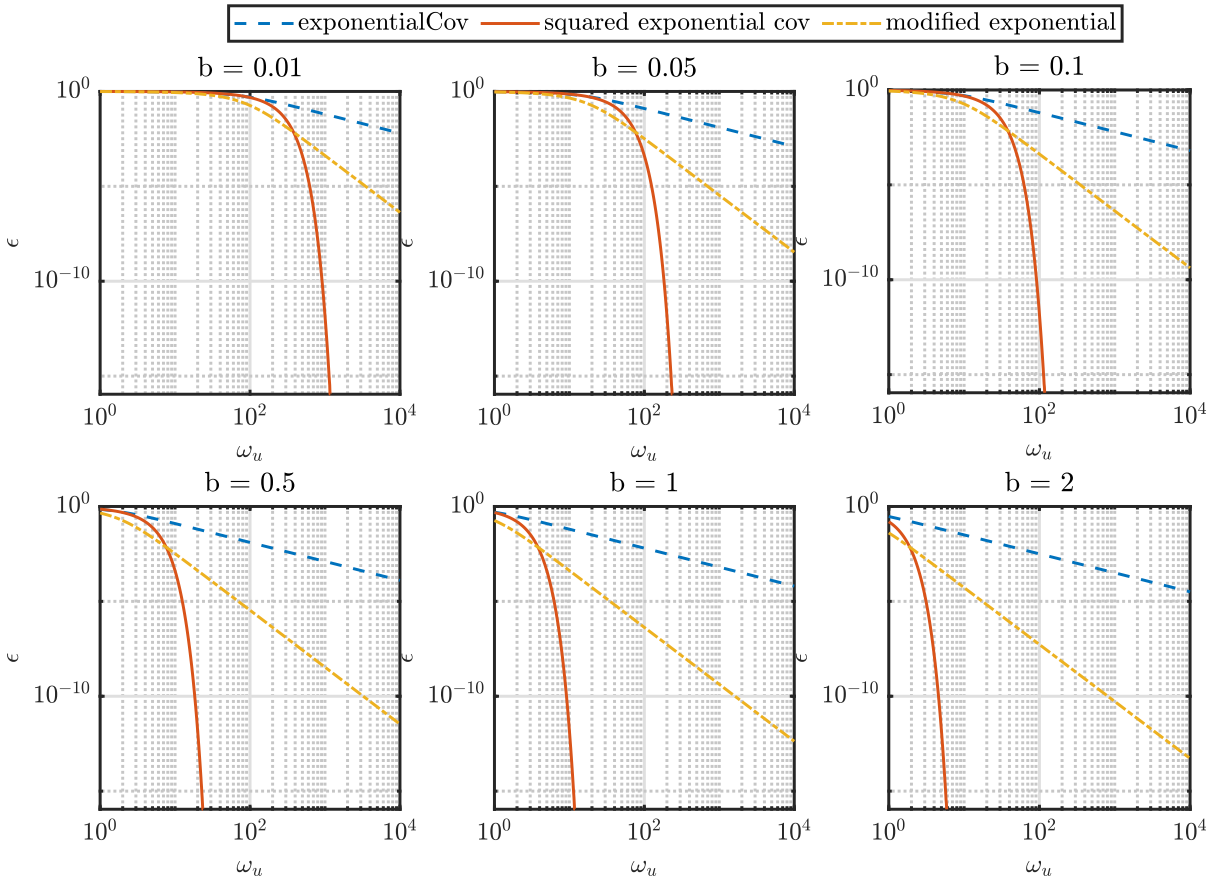

Figure 1. Decay of the energy approximation error e as a function of the truncation frequency ω_u.

by means of substitution and given for these power spectra by the equations

$$\int_0^\infty S_{ff}^s(\omega)d\omega = \left[\frac{\sigma^2}{\pi}\tan^{-1}(b\omega)\right]_0^\infty = \frac{\sigma^2}{2}, \tag{24}$$

$$\int_0^\infty S_{ff}^m(\omega)d\omega = \left[\frac{\sigma^2}{\pi}4b\left(\frac{\omega}{2(\omega^2b^2+1)} + \frac{1}{2b}\tan^{-1}(b\omega)\right)\right]_0^\infty = \sigma^2, \tag{25}$$

and

$$\int_0^\infty S_{ff}^{sq}(\omega)d\omega = \left[\sigma^2\text{erf}\left(\frac{b\omega}{2}\right)\right]_0^\infty = \sigma^2, \tag{26}$$

where $\text{erf}(x) = \frac{1}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ is the error function.

Similarly, the left-hand side of Eq. (13) for the considered autocorrelation functions can be determined as

$$\begin{aligned}\int_0^{\omega_u} S_{ff}^s(\omega)d\omega &= \left[\tfrac{\sigma^2}{\pi}\tan^{-1}(b\omega)\right]_0^{\omega_u} \\ &= \tfrac{\sigma^2}{\pi}\tan^{-1}(b\omega_u),\end{aligned} \tag{27}$$

$$\begin{aligned}\int_0^{\omega_u} S_{ff}^m(\omega)d\omega &= \left[\tfrac{\sigma^2}{\pi}4b\left(\tfrac{\omega}{2(\omega^2b^2+1)} + \tfrac{1}{2b}\tan^{-1}(b\omega)\right)\right]_0^{\omega_u} \\ &= \tfrac{\sigma^2}{\pi}4b\left(\tfrac{\omega_u}{2(\omega_u^2b^2+1)} + \tfrac{1}{2b}\tan^{-1}(b\omega_u)\right),\end{aligned} \tag{28}$$

and

$$\begin{aligned}\int_0^{\omega_u} S_{ff}^{sq}(\omega)d\omega &= \left[\sigma^2\text{erf}\left(\tfrac{b\omega}{2}\right)\right]_0^{\omega_u} \\ &= \sigma^2\text{erf}\left(\tfrac{b\omega_u}{2}\right).\end{aligned} \tag{29}$$

Using these equations in conjunction with Eq. (13), and isolating the desired truncation error e_S provides relations between the error and the cut-off frequency as. Specifically,

$$e_{S,s} = 1 - \frac{2}{\pi}\tan^{-1}(b\omega_u), \tag{30}$$

$$e_{S,m} = 1 - \frac{2}{\pi}\left(\frac{b\omega_u}{b^2\omega_u^2+1} + \tan^{-1}(b\omega_u)\right), \tag{31}$$

and

$$e_{S,sq} = 1 - \text{erf}\left(\omega_u\frac{b}{2}\right). \tag{32}$$

Figure 1 shows the decay of the truncation error of the energy e_S as a function of the selected ω_u for the considered autocorrelation models, for several different values of the correlation lengths $b = [0.01, 0.05, 0.1, 0.5, 1, 2]$. The decay function is calculated according to Eq. (30) to (32). In particular,

a zero-mean Gaussian stochastic process with unit variance on a domain $t \in [0, 5]$ is considered for this purpose. From this Figure, it is clear that this approximation error decays significantly faster in squared and modified exponential kernels as compared to a single exponential kernel. Specifically, the squared exponential kernel converges super-linearly, the modified exponential kernel converges linearly with a slope of 2.13 and the single exponential kernel converges linearly with a slope of 0.87. These findings are valid for all correlation lengths tested. The difference in the convergence rate between the modified and single exponential kernel originates from the $\frac{b\omega_u}{b^2\omega_u^2+1}$-term in Eq. (31). This term ensures that the error for the modified exponential kernel decreases faster versus the single exponential case when $w_u \to \infty$. However, note that the shape of the decay of the error with respect to ω_u is similar for both single and modified exponential curves.

In Figure 2 sample paths of the corresponding Gaussian stochastic processes are shown for different values of the truncation error e_S. Each sample path is created with all the three correlation structures in the time $t \in [0, 5]$, but only $t \in [0, 0.1]$ is shown to better visualize the local characteristic of the samples. This Figure shows how the sample path from the exponential correlation exhibits non-negligible high frequency oscillations and noise when compared with the other two correlations when lower values of e_S are used. This is a direct result of the slower convergence of the single exponential kernel in energy content, which in it's turn is related to the non-differentiability of R_{ff} at zero lag. As pointed out in the literature, these high-frequency oscillations potentially impact the engineering analysis affected by the properties modeled with this kernel.

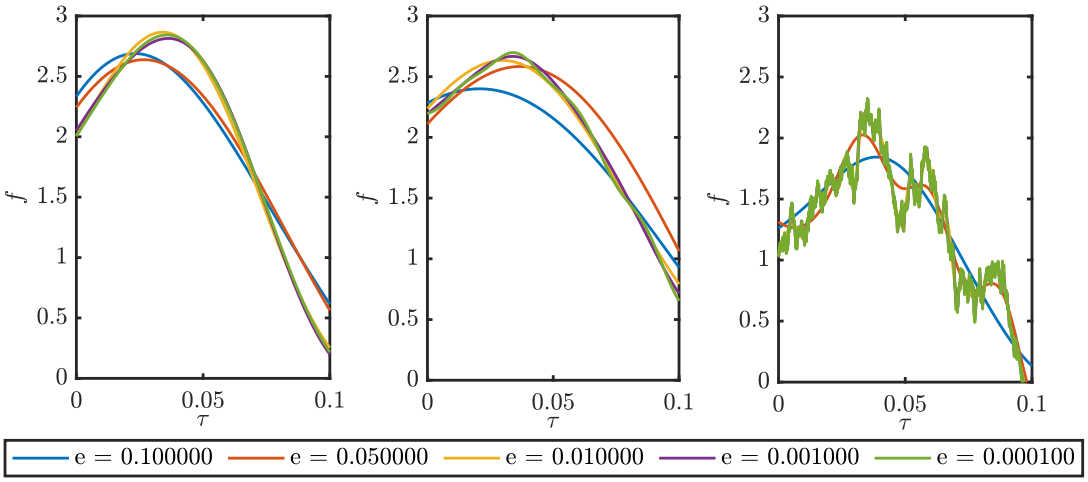

Figure 2. Sample paths of a one-dimensional Gaussian stochastic process, where the autocorrelation is governed by a Squared exponential (left), Modified exponential (middle) and Single exponential (right) kernel function, for different values of the truncation error.

3.2. Karhunen-Loève expansion

The advantage of using the modified exponential kernel over a single exponential one in terms of required terms in the Karhunen-Loève expansion has been illustrated in (Spanos, Beer and Red-Horse, 2007). In this section, additional insights into this topic are given. In particular, it is shown

that the reduction in required number of stochastic terms holds consistently for the entire process series expansion. Further, this property also holds for a wide range of correlation length values. Figure 3 shows the convergence of the mean error variance as a function of the number of terms that are retained in the KL expansion for the same stochastic process considered in Section 3.1. In this paper, the solution of the Fredholm integral equation in Eq. 16 is obtained using a Galerkin procedure with Legendre basis functions. Particularly the solution of the cases where more than 1000 terms are requirede were found to be computationally non-trivial. Clearly, the convergence of the mean error variance corresponding to the squared exponential process is the fastest of the three considered autocorrelation functions. These results further confirm the findings of (Spanos, Beer and Red-Horse, 2007), namely that the modified exponential kernel converges faster than the single exponential kernel when considering the KL expansion for a wide range of correlation length values. This can be also explained by the differentiability of the autocorrelation function at zero-lag.

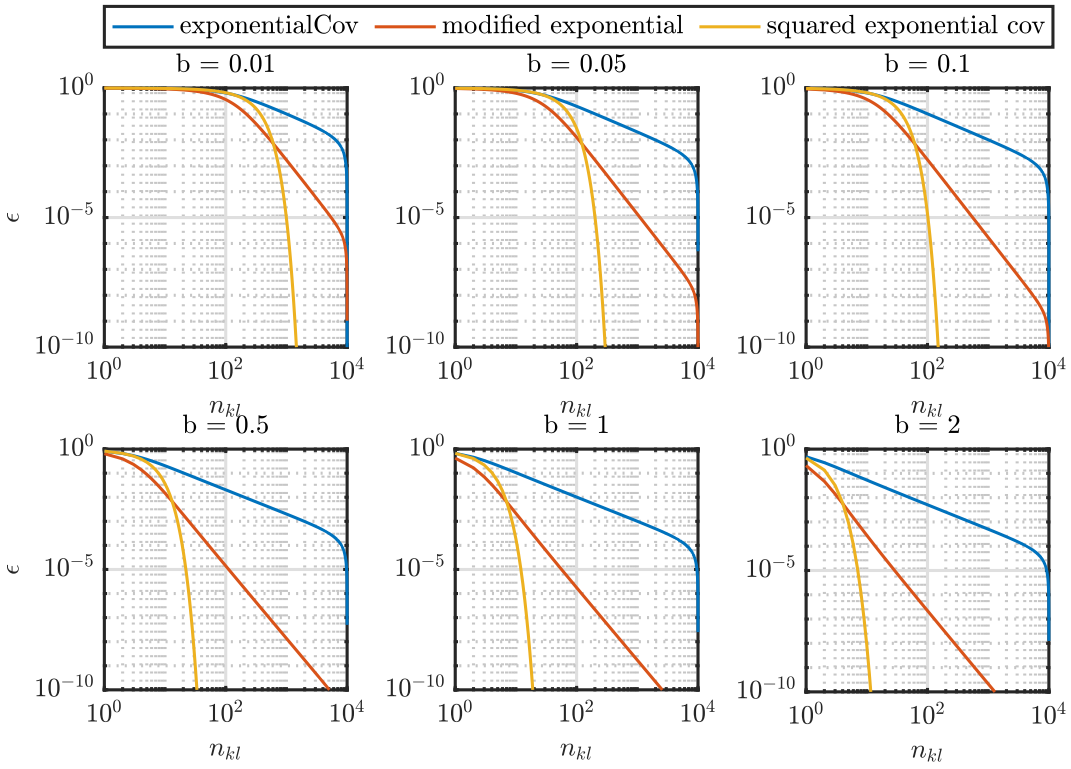

Figure 3. Convergence of the mean error variance with respect to the number of terms in the KL Expansion for the three considered autocorrelation kernels.

3.3. AR model representation

For the study of the applicability of the considered autocorrelation processes with an AR representation, an analytical approach is used. In particular, the required AR models are derived based on the considered auto-correlation functions.

Concerning the single exponential covariance kernel $R^s_{ff}(\tau)$, it can be shown that an AR(1) process, i.e., an AR process with $m = 1$ is capable of directly representing a Gaussian process with $R^s_{ff}(\tau)$ (Vanmarcke, 1983). To capture this, write out Eq. (20) for $m = 1$ and subtract $f(t, \theta)$ from both sides. That is,

$$[f(t + 1, \theta) - f(t, \theta)] + (1 - a)f(t, \theta) = b_0 w(t, \theta). \tag{33}$$

The associated first-order differential equation to this finite-difference equation is

$$\frac{d}{dt} f(t, \theta) + \alpha f(t, \theta) = b_0 w(t, \theta), \tag{34}$$

which can be interpreted as a linear model that links $f(t, \theta)$ to a white-noise excitation. In the limit state for infinitesimally small τ, the transfer function corresponding to Eq.(34) is

$$H(\omega) = \frac{1}{i\omega + \alpha}, \tag{35}$$

that in combination with a constant spectral density S_0 gives rise to the power spectral density $S_{ff}(\omega)$:

$$S_{ff}(\omega) = \frac{S_0}{\omega^2 + \alpha^2}, \tag{36}$$

for $f(, \omega)$. This can be related through Eq. (6) to the autocorrelation function:

$$R_{ff}(\tau) = \sigma^2 \exp(-|\tau|\alpha), \tag{37}$$

which corresponds to a single exponential autocorrelation function.

Following a similar procedure, it can be shown that an AR(1) model of the following form:

$$f(t, \theta) = a\,[f(t - 1, \theta) + f(t + 1, \theta)] + b_o 2(t, \theta) \tag{38}$$

corresponds to a stationary Gaussian process with power spectral density:

$$S_{ff}(\omega) = \frac{4\alpha^3 \sigma^2}{\pi(\omega^2 + \alpha^2)^2}, \tag{39}$$

and hence with auto correlation function

$$R_{ff}(\tau) = \sigma^2 \exp(-\alpha|\tau|)(\alpha|\tau| + 1), \tag{40}$$

that corresponds to the modified exponential kernel.

For the squared exponential autocorrelation function, a one-on-one exact relationship with a corresponding AR model is not trivial to establish and requires the solution of the Yule-Walker equations. This falls outside the scope of this paper.

4. Concluding remarks

In this paper, certain additional aspects relating to the mathematical behavior of common autocorrelation functions have been studied. Specifically, the convergence of the spectral density of the process to zero as the frequency tends to infinity has been examined. In particular, attention has been focussed on the comparison of the spectral convergence of functions that are differentiable at zero-lag versus those that are not-differentiable. It has been shown, both analytically and numerically, that the number of stochastic components required to represent a stochastic process with the single exponential kerne is considerably larger when compared to a modified or squared exponential kernel. This statement holds for both stochastic spectral representation methods, as well as for the well-known Karhunen-Loève series expansion. In this context, it must be noted that the single exponential kernel is not differentiable at zero lag, whereas, the modified and squared exponential kernel are completely differentiable. Further, in the paper AR models have been considered, for which it has been shown that for both a single and modified exponential kernel, a closed form expression for an AR(1) model can be derived. This proves that AR models are highly suited to represent these types of stochastic processes. Further work beyond this initial study will explore the relationship between the differentiability of the autocorrelation function at zero lag, and the convergence of the corresponding spectral density to zero as the frequency tends to infinity.

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