Random set solutions to partial differential equations

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Abstract. The past decades have seen increasing interest in modelling uncertainty by heterogeneous methods, combining probability and interval analysis, especially for assessing parameter uncertainty in engineering models. A unifying mathematical framework admitting the combination of a wide range of such methods is the theory of random sets, describing input and output of a structural model by set-valued random variables. The purpose of this paper is to highlight the mathematics behind this approach. The modelling and computational implications are discussed and demonstrated with the help of a numerical example from elastostatics.

Keywords: Partial differential equations; random fields; interval parameters; random sets

1. Introduction

In many branches of engineering, such as linear elasticity or wave propagation, the quantities of interest are solutions to equilibrium equations or to equations of motion. These equations contain coefficients that—as a rule—depend on further parameters, such as material constants or geometric dimensions. In addition, the parameters may vary in space and time. A major task in engineering design is the analysis of the uncertainty of the computed output. This uncertainty traces back to uncertainty in the input parameters, which may be due to many sources: random fluctuations, lack of information, conflicting assessments, random measurement errors, systematic measurement errors, fluctuations due to spatial inhomogeneity, errors made by assigning parameter status to state variables, model insufficiency, just to name a few. (Uncertainty in the model itself is important, but will be left aside in this paper.)

Available information on parameter variability may consist in frequency distributions obtained from large samples, values from small samples or single measurements, interval bounds, experts' point estimates, or educated guesses from experience. These different types of uncertainty have often been classified into uncertainties of aleatoric or of epistemic type. Accordingly, an increasing desire has risen in the engineering community to introduce methods of quantifying the uncertainty beyond probability, such as interval analysis, set-valued models, fuzzy sets, evidence theory, random sets, sets of probability measures, imprecise probability, lower and upper previsions, info-gap-analysis, etc. This paper focusses on combinations of probability and interval analysis. This includes random variables, random fields, intervals, interval fields, random variables with interval parameters and random fields with interval parameters (such as mean, variance, correlation length).

All these types and their combinations can be accommodated in the framework of *random sets*. Random sets are set-valued random variables, whereby the values may be sets of real numbers,

subsets of finite dimensional vector spaces, or subsets of function spaces. For example, random variables are single-valued random sets and intervals are random sets on a one-point probability space. The purpose of this paper is to highlight some important mathematical modelling questions that arise when using random sets. The considerations will be exemplified with the help of a static model, the equilibrium equation $-\operatorname{div}(a(x)\operatorname{grad} u(x)) = f(x)$ for the transversal displacement $u(x)$ of a membrane. The coefficient $a(x)$ contains the uncertain material properties of the membrane, while $f(x)$ is the load, both of which may vary with respect to location x. When $a(x)$ is a random field, a random field with interval parameters, or a (finite) interval field, the resulting solution $u(x)$ is no longer single-valued, but rather a random set with values in a function space. (The values at fixed points x are real-valued random sets.) This situation has been investigated from several viewpoints, including computational aspects, in the literature; see e.g. (Do, Gao, Song and Beer, 2016; Faes and Moens, 2017; Muscolino, Santoro and Sofi, 2016; Oberguggenberger, 2015; Oberguggenberger and Wurzer, 2019; Schmelzer, Oberguggenberger and Adam, 2010; Sofi and Romeo, 2018). Papers laying special emphasis on comparing stochastic and interval methods are (Gao, Song and Tin-Loi, 2010; Jiang, Zheng and Han, 2018; Muscolino, Sofi and Zingales, 2013; Sofi, 2015; Sofi, Muscolino and Giunta, 2020).

In order to establish that the values of the solution are intervals, random variables, or random sets, the main issue is the continuous dependence of the solution on the parameters of the random field. The arguments are rather straightforward in the case of intervals or random variables/fields, but involve some subtleties when the latter are combined, resulting in general random sets.

The plan of the paper is as follows. Section 2 will be devoted to the required mathematical set-up. These facts are known, but it appears useful to elaborate them in some detail for the later applications. Section 3 singles out the principal mathematical argument that allows one to prove that the output quantity is a random set. The combination of arguments giving a full proof of this result seems to be new. Section 4 details the considerations for the membrane equation and is a new application of the findings of Section 3. Section 5 contains a comparison with the parametric approach and numerical aspects. Detailed proofs can be found in the dissertation of the first author (Nedeljković, 2020).

2. Mathematical preliminaries

This section specifies some mathematical notions which are required for a rigorous treatment of random sets. Recall that a subset S of a finite dimensional vector space such as n-dimensional Euclidean space $\mathbb{R}^d$ is *open*, if for each point x in S, each ball of the form $\{y \in \mathbb{R}^d : \|y - x\| < \eta\}$ with center in x and sufficiently small radius η is also contained in S. The set S is *closed*, if with each converging sequence whose terms belong to S, the limit also belongs to S. The subset S is *bounded*, if the norms $\|x\|$ of the elements of S are all smaller than a fixed constant. Further, finite dimensional Euclidean spaces are *separable*, that is, they contain a countable dense subset (a sequence of points such that every ball in $\mathbb{R}^d$ contains at least one element of the sequence), and they are *complete* (that is, if x_k is a sequence such that the sum of its norms $\sum_{k=1}^{\infty} \|x_k\|$ converges, then the sum $\sum_{k=1}^{\infty} x_k$ also converges).

All these notions can be literally generalized to infinite dimensional normed spaces. A typical example is the space of continuous functions on a bounded, closed interval I, denoted by $\mathcal{C}(I)$, with norm $\|f\| = \max_{x\in I} |f(x)|$. This space is separable and complete.

2.1. Random variables

A real-valued random variable a is characterized by its distribution function $F_a(x)$ which defines the probabilities P of basic events of the form $\{a \leq x\}$ via $P(a \leq x) = F_a(x)$. This point of view suffices for most purposes. However, in the sequel *families of random variables* will be needed, and it will be necessary to specify the domains of the random variables under consideration. The usual mathematical setting is a probability space (Ω, Σ, P) where Ω is a set (the collection of all elementary events), Σ is the family of measurable subsets of Ω, and P is a probability measure (that assigns to every set belonging to Σ a number between 0 and 1, satisfying the usual axioms). In this point of view, a random variable is a mapping from Ω into the real line such that the events $\{\omega \in \Omega : a(\omega) \leq x\}$ are measurable (i.e., belong to Σ). Thus a probability $P(\{\omega \in \Omega : a(\omega) \leq x\})$ can be assigned to such events, denoted again by $P(a \leq x)$, and these probabilities in turn define the distribution function of the random variable a. From the probabilities $P(a \leq x)$, the probabilities of open and closed intervals, their complements, their countable unions and intersections etc. can be computed. The family of subsets of the real line which can be obtained by indefinite repetition of these set theoretic operations defines the so-called *Borel σ-algebra* $\mathcal{B}$. A random variable has the property that all sets of the form $\{\omega \in \Omega : a(\omega) \in B\}$, where B is any Borel set, are measurable. Accordingly, random variables are often referred to as measurable functions.

If the probability space is finite, say $\Omega = \{1, \ldots, m\}$, one may take $\Sigma = \Omega$, and every real-valued function on Ω is measurable. A typical infinite probability space is standard Gaussian space with $\Omega = \mathbb{R}$, Σ the Borel measurable subsets of $\mathbb{R}$, and the probability of an event B given by the Gaussian integral $P(B) = (2\pi)^{-1/2} \int_B \mathrm{e}^{-\omega^2/2} \mathrm{d}\omega$; the probability measure P is given through the Gaussian density $\mathrm{e}^{-\omega^2/2}/\sqrt{2\pi}$. On standard Gaussian space, every continuous real-valued function on Ω is a random variable.

These notions can be easily extended to finite dimensional random variables, i.e., with values in $\mathbb{R}^d$ (then the joint probability distributions have to be specified). Measurable functions with values in an infinite dimensional normed space can be defined in the same way.

2.2. Random fields

A random field is a process that assigns a random variable $q(x)$ to every point x in a region D in space. To define the probabilistic properties of the field, the joint distributions of the values at any finite number of points $q(x_1), \ldots q(x_n)$ should be specified. If the random field is Gaussian, it is completely specified by the mean value $\mu_q = \mathrm{E}(q(x))$ and the second moments, i.e., the covariance $\mathrm{COV}(q(x), q(y))$ for any two points x, y. If it is (weakly) homogeneous and isotropic, the covariance depends only on the distance $\rho = \|x - y\|$ of the points and is of the form $\mathrm{COV}(q(x), q(y)) = C(x, y) = \sigma^2 c(\rho)$ with the field variance σ^2 and the autocorrelation function

$c(\rho)$. A typical autocorrelation function is of the form

$$c(\rho) = \exp\Big(-\frac{|\rho|}{\ell}\Big), \tag{1}$$

where ℓ is the so-called correlation length. The standard method of simulation of a random field is based on the *Karhunen-Loève expansion*. It is obtained by solving the eigenvalue problem

$$\int_D C(x,y)\varphi_k(y)\,\mathrm{d}y = c_k\varphi_k(x) \tag{2}$$

which has a sequence of positive eigenvalues c_k and orthonormal eigenfunctions $\varphi_k(x)$ (orthonormality in mean square). Then

$$q(x) = \sum_{k=1}^{\infty} \sqrt{c_k}\xi_k\varphi_k(x) \tag{3}$$

where the ξ_k are uncorrelated random variables with unit variance. If the process is Gaussian, the ξ_k are independent and distributed according to $\mathcal{N}(0,1)$.

For the numerical simulation, the spatial region is discretized by a grid and the φ_k are taken, e.g., piecewise constant on the grid elements. The eigenvalue problem becomes a matrix eigenvalue problem, and the series (3), with approximate eigenvalues and eigenfunctions, truncated after a finite number M of terms, can be used for Monte Carlo simulations of the field trajectories (i.e., realizations of the field).

If the random field $q(x)$ is the input, say, in a static structural model, the model output $u(x)$ is also a random field. The required number of terms M in the expansion depends crucially on the correlation length and the smoothness of the input-output function $q \to u$. The model output in turn depends, among others, on the field parameters μ, σ, ℓ. When applying random sets subsequently, information about the type of dependence on these parameters (e.g., continuity) is needed. This is aided by the fact that a mean zero random field on $D = \mathbb{R}$ with autocorrelation function (1) and variance σ^2 can be obtained as an Ornstein-Uhlenbeck process, namely as solution to the Langevin stochastic differential equation

$$\mathrm{d}q = -\frac{1}{\ell}q\,\mathrm{d}x + \sqrt{\frac{2}{\ell}}\,\sigma\,\mathrm{d}W, \quad q|_{x=0} \sim \mathcal{N}(0,\sigma^2), \tag{4}$$

where $W = W(x,\omega)$ denotes Wiener process on the real line. In this case, realizations of the random field can be generated by simulating solutions to the Langevin equation (Arnold, 1974).

2.3. Random sets

In general, a *random set* is a set-valued random variable satisfying certain measurability conditions, to be detailed below. The simplest case arises when the underlying probability space is finite. In this case, one speaks of *finite random sets* or *Dempster-Shafer structures*. Such a structure is given by finitely many closed subsets $A_i, i = 1,\dots,m$ of a target space $\mathcal{A}$, usually Euclidean space $\mathbb{R}^d$, called the *focal elements*, each of which comes with a *probability weight* $p_i = p(A_i)$, $\sum p_i = 1$.

Viewed as a random set, a Dempster-Shafer structure is given by an m-point probability space $\Omega = \{1, 2, \ldots, m\}$ with probability masses $\{p_1, p_2, \ldots, p_m\}$; the assignment $i \to A_i$ is the defining set-valued random variable A.

However, random sets generated by infinite families of random variables will be needed in the context of random fields. This leads to the notion of infinite random sets. In this case, the random sets will be defined on a general probability space (Ω, Σ, P). Also, the target space can be infinite-dimensional, for example, a function space containing the system outputs.

In the sequel, the target space will be either Euclidean space $\mathbb{R}^d$ or more generally a normed, complete and separable space $\mathcal{A}$. Given a set-valued map $\omega \to A(\omega)$ where each $A(\omega)$ is a subset of the target space $\mathcal{A}$, upper and lower inverses of subsets B of the target space $\mathcal{A}$ are defined by

$$A^-(B) = \{\omega \in \Omega : A(\omega) \cap B \neq \emptyset\}, \qquad A_-(B) = \{\omega \in \Omega : A(\omega) \subset B\}. \tag{5}$$

The requirements that $\omega \to A(\omega)$ is a *random set* are

- each $A(\omega)$ is closed;
- the upper inverses $A^-(B)$ are measurable subsets of Ω for every closed subset B of the target space $\mathcal{A}$.

Recall that a *measurable selection* of a random set A is a random variable a such that $a(\omega) \in A(\omega)$ for all $\omega \in \Omega$. An important tool is the *fundamental measurability theorem*, see e.g. (Molchanov, 2005), that states the equivalence of the measurability property of $A^-(B)$ for open, for closed, and for Borel subsets B of the target space as well as the equivalence with the existence of a *Castaing representation*. A Castaing representation is a sequence of measurable selections $a_k(\omega) \in A(\omega)$ such that for all ω, $\{a_k(\omega) : k = 1, 2, 3, \ldots\}$ is dense in $A(\omega)$. (To be precise, the fundamental measurability theorem requires *completeness of the probability space* (Ω, Σ, P) in the sense that every subset of a set of probability zero belongs to Σ.)

The lower inverses of events B can be written as $A_-(B) = \{\omega \in \Omega : A(\omega) \cap B^{\mathsf{c}} \neq \emptyset\}^{\mathsf{c}}$, where B^{c} denotes the complement of B, and hence are measurable due to the fundamental measurability theorem and the measurability of $A^-(B)$. Thanks to the required measurability properties, it is legitimate to introduce upper and lower probabilities

$$\overline{P}(B) = P(A^-(B)) = P(\{\omega \in \Omega : A(\omega) \cap B \neq \emptyset\}), \quad \underline{P}(B) = P(A_-(B)) = P(\{\omega \in \Omega : A(\omega) \subset B\})$$

for any Borel set $B \subset \mathcal{A}$.

The concept of random sets encompasses many other well-established methods of uncertainty modelling. Clearly, every interval defines a random set (as a random variable on a one-point probability space). Similarly, every random variable a on a probability space (Ω, Σ, P) defines a (single-valued) random set $\omega \to A(\omega) = \{a(\omega)\}$. The upper and lower probability of an event B coincide with its probability $P(B)$. Further, every normalized fuzzy number can be viewed as a random set. Here the probability space is the interval $\Omega = [0, 1]$, equipped with the uniform probability distribution, and the focal elements $A(\omega)$ are just the ω-level sets. It is not difficult to prove that the possibility measure $\pi(B)$ of a subset B of the real line coincides with its upper probability (Goodman and Nguyen, 2002). Indeed, and rather obviously,

$$\pi(B) = \sup_{\omega \in [0,1]} \{A(\omega) \cap B \neq \emptyset\} = P(\{\omega \in \Omega : A(\omega) \cap B \neq \emptyset\}) = \overline{P}(B).$$

2.4. Random sets generated by families of random variables

To set the stage, consider random variables a_λ, defined on the same probability space (Ω, Σ, P), and depending on a set of parameters $\lambda \in \Lambda$. Suppose the random variables have values in a target space $\mathcal{A}$. Then one can define a set-valued map $\omega \to A(\omega) = \{a_\lambda(\omega) : \lambda \in \Lambda\}$. The question is whether this assignment defines a random set. Before answering this question, consider the example of an imprecise Gaussian family

$$a_\lambda(\omega) = \mu + \sigma\Phi^{-1}(\omega) \tag{6}$$

with interval parameters $\lambda = (\mu, \sigma) \in \Lambda = [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$. Here the underlying probability space is the unit interval $\Omega = (0, 1)$ with the uniform distribution, and Φ is the standard normal distribution function. It follows that each random variable a_λ is distributed according to $\mathcal{N}(\mu, \sigma^2)$. The corresponding subset of the real line is

$$A(\omega) = [\underline{a}(\omega), \overline{a}(\omega)] \quad \text{with} \quad \underline{a}(\omega) = \min\left(a_\lambda(\omega) : \lambda \in \Lambda\right), \quad \overline{a}(\omega) = \max\left(a_\lambda(\omega) : \lambda \in \Lambda\right).$$

Due to the fact that the parameter set Λ is a (two-dimensional) closed and bounded interval and the fact that the assignment $(\mu, \sigma) \to \mu + \sigma\Phi^{-1}(\omega)$ is continuous at fixed ω, it follows that each $A(\omega)$ is a closed and bounded interval, indeed. Some members of the family and the resulting random intervals are depicted in Figure 1. Anticipating that $\omega \to A(\omega)$ is a random set, one may define the

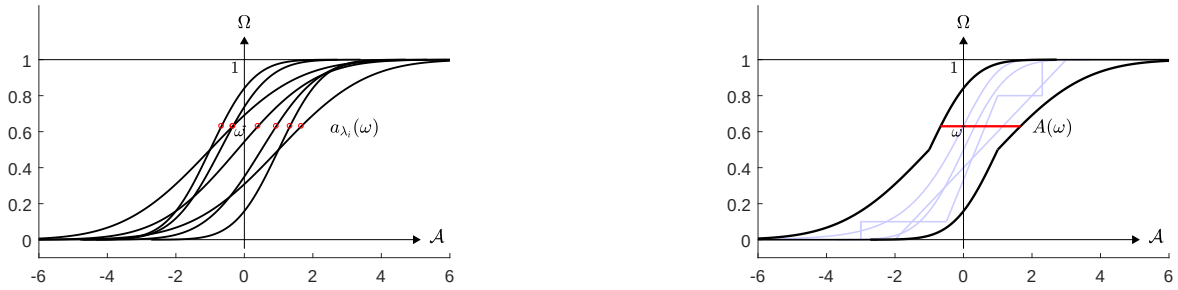

Figure 1. Some members of the family a_λ (left), random set with focal element (right). The parameters $\lambda = (\mu, \sigma)$ were taken as $-1 \le \mu \le 1$, $1 \le \sigma \le 2$.

upper and lower distribution functions

$$\underline{F}(b) = P(\omega : \overline{a}(\omega) \le b), \quad \overline{F}(b) = P(\omega : \underline{a}(\omega) \le b).$$

The upper and lower distribution functions enclose a so-called *probability box* or p-box (Ferson, Ginzburg, Kreinovich, Myers and Sentz, 2003). Note that the p-box contains many probability distribution functions other than the one coming from the Gaussian family. Anticipating also that the functions $\underline{a}$, $\overline{a}$ bounding the random intervals are random variables, one obtains simply that $\underline{F}$ is the distribution function of $\overline{a}$ and $\overline{F}$ is the distribution function of $\underline{a}$.

Here is the argument why $\omega \to A(\omega)$ is a random set. First, one constructs a Castaing representation by taking a countable dense subset of parameter values $\lambda_k = (\mu_k, \sigma_k)$ in $\Lambda = [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}]$, for example, all points with rational coordinates. Again from the continuity of the assignment $\lambda \to$

$a_\lambda(\omega)$ at fixed ω, i.e., $(\mu, \sigma) \to \mu + \sigma\Phi^{-1}(\omega)$, see (6), it follows that the set $\{a_{\lambda_k}(\omega) : k = 1, 2, 3, \ldots\}$ is dense in $A(\omega)$ for every fixed ω. Let B be an open set. Then

$$\begin{aligned} A^-(B) &= \{\omega \in \Omega : A(\omega) \cap B \neq \emptyset\} = \{\omega \in \Omega : \text{ there is } k \text{ such that } a_{\lambda_k}(\omega) \in B\} \\ &= \bigcup_k \{\omega \in \Omega : a_{\lambda_k}(\omega) \in B\} \end{aligned} \tag{7}$$

is measurable as a countable union of measurable sets. (The individual sets are measurable, because the maps $\omega \to a_\lambda(\omega)$ are measurable at each fixed λ.) The fundamental measurability theorem implies that $A^-(B)$ is measurable for every Borel set B, in particular, for every closed set B. This is the required property for $\omega \to A(\omega)$ to be a random set.

Using again that $\{a_{\lambda_k}(\omega) : k = 1, 2, 3, \ldots\}$ is dense in $A(\omega)$, it follows that $\underline{a}(\omega) = \inf\{a_{\lambda_k}(\omega) : k = 1, 2, 3, \ldots\}$ and $\overline{a}(\omega) = \sup\{a_{\lambda_k}(\omega) : k = 1, 2, 3, \ldots\}$ are measurable as infima and suprema of countably many random variables.

It is seen that the required arguments are quite elaborate. The underlying structure will be worked out in the following sections. Further details on interpretations and applications of random sets can be found, e.g., in the monographs (Bernardini and Tonon, 2010; Molchanov, 2005; Nguyen, 2006) and the survey papers (Beer, Ferson and Kreinovich, 2013; Oberguggenberger, 2013).

3. Random sets generated by parametrized random fields

This central section lays down the basic structure that will allow one to prove that set-valued maps obtained as solutions to partial differential equations whose coefficients are given as intervals, random fields, or parametrized random fields, are indeed random sets.

3.1. The general measurability result

For the present purpose, the most general situation to be considered is the following.

- (Ω, Σ, P) is a complete probability space;
- the target space $\mathcal{A}$ is a normed, complete and separable space;
- Λ is a bounded and closed subset of some Euclidean space;
- $a_\lambda(\omega)$, $\lambda \in \Lambda$ is a family of random variables with values in $\mathcal{A}$, that is, the maps $\omega \to a_\lambda(\omega)$ are measurable for each λ;
- the random variables depend continuously on λ, that is, the maps $\Lambda \to \mathcal{A} : \lambda \to a_\lambda(\omega)$ are continuous for each ω.

Then the following assertions hold: The set-valued map

$$\omega \to A(\omega) = \{a_\lambda(\omega) : \lambda \in \Lambda\}$$

defines a random set. All focal elements $A(\omega)$ are bounded and closed subsets of $\mathcal{A}$. In addition, if Λ is a (multi-dimensional) interval and $\mathcal{A} = \mathbb{R}$, the sets $A(\omega)$ are intervals.

Proof of assertion. First, it is clear from the continuity assumption that all $A(\omega)$ are bounded and closed subsets of $\mathcal{A}$. The assertion that they are intervals under the additional specializing hypothesis follows from the continuity as well. To prove the random set property, one has to show that the upper inverses $A^-(B)$ are measurable for every open subset B of $\mathcal{A}$. This follows exactly by the argument given in equation (7) and applying the fundamental measurability theorem.

Example. Consider a function $u = u(\lambda, a)$ that depends on an interval parameter λ, say $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, and a fixed random variable a. Such a situation arises, for example, when u is the response of a system with mixed interval and random uncertainty. Both λ and a can be multi-dimensional, so this covers the case of discretized interval and random fields. Assume that the dependence on λ and a is continuous. The functions $a_\lambda(\omega) = u(\lambda, a(\omega))$ form a family of random variables exactly of the type discussed here. Collecting the responses in $A(\omega) = \{u(\lambda, a(\omega)) : \lambda \in \Lambda\}$ therefore leads to a random set, actually a random interval $\omega \to [\underline{a}(\omega), \overline{a}(\omega)]$ if the response is one-dimensional.

3.2. Continuous dependence of random fields on their parameters

A more involved instance of a random set is obtained by taking a random field on the real line ($x \in \mathbb{R}$) whose parameters are intervals. For example, consider mean zero random fields $q(x, \omega, \ell)$ with fixed variance σ^2, but correlation length ℓ varying in an interval $[\underline{\ell}, \overline{\ell}]$. A common probability space for these random fields with different correlation lengths is needed. A convenient way of generating such random fields, in the case of the autocorrelation function (1), is by means of the Langevin equation (4). The common probability space is Wiener space Ω_W (the space of trajectories of Wiener process with the induced probability measure). Due to the fact that the random fields $q(x, \omega, \ell)$ are solutions to (4), the following properties hold; see e.g. (Schmelzer, 2010; Schmelzer, 2013):

- For fixed $x \in \mathbb{R}$ and $\ell \in [\underline{\ell}, \overline{\ell}]$, the map $\omega \to q(x, \omega, \ell)$ is measurable.
- For fixed $x \in \mathbb{R}$ and $\omega \in \Omega_W$, the map $\ell \to q(x, \omega, \ell)$ is continuous.

Thus at every chosen point x, the assignment $\omega \to Q(x, \omega) = \{q(x, \omega, \ell) : \ell \in [\underline{\ell}, \overline{\ell}]\}$ defines a (scalar) random set, as can be seen by the same arguments as in the example above. The trajectories of the random field are interval-valued curves, given by $x \to [\min(q(x, \omega, \ell), \max(q(x, \omega, \ell)]$, where the minimum and maximum are obtained by letting ℓ run through $[\underline{\ell}, \overline{\ell}]$.

Alternatively, the Karhunen-Loève expansion (3) can be used to place the random fields $q(x, \omega, \ell)$ in a common probability space Ω, not depending on ℓ. Formally, this can be done by considering an infinite product of standard Gaussian spaces with elements $\omega = (\omega_1, \omega_2, \omega_3, \ldots)$. The required sequence of Gaussian variables ξ_k is then given by $\xi_k(\omega) = \omega_k$, and the Karhunen-Loève expansion reads

$$q(x, \omega, \ell) = \sum_{k=1}^{\infty} \sqrt{c_{k,\ell}}\, \xi_k(\omega) \varphi_{k,\ell}(x). \tag{8}$$

In general, there is no way to assert that the eigenvalues $c_{k,\ell}$ and eigenfunctions $\varphi_{k,\ell}(x)$ depend continuously on the correlation length ℓ, even after discretization and reduction to a matrix eigenvalue problem. Thus one has to analyze the situation in each case individually. The case of the autocovariance function (1) on a finite interval can be treated explicitly, thanks to the results in

(Ghanem and Spanos, 1991). Without restriction of generality, one may assume that the finite interval is $D = [-1, 1]$. Then the eigenvalue problem (2) becomes

$$\int_{-1}^{1} \mathrm{e}^{-|x-y|/\ell} \varphi(y)\, \mathrm{d}y = c\varphi(x). \tag{9}$$

Following (Ghanem and Spanos, 1991), problem (9) can be solved in the following way. First, the equations

$$\frac{1}{\ell} - \alpha \tan \alpha = 0, \quad \text{respectively} \quad \alpha + \frac{1}{\ell} \tan \alpha = 0 \tag{10}$$

have two sequences of positive solutions denoted by $\alpha_k = \alpha_k(\ell)$ and $\alpha_k^* = \alpha_k^*(\ell)$, for even and odd k, respectively. The resulting eigenvalues and eigenfunctions are

$$c_{k,\ell} = \frac{2\ell}{1 + \ell^2 \alpha_k^2(\ell)},\ c_{k,\ell}^* = \frac{2\ell}{1 + \ell^2 \alpha_k^{*2}(\ell)},\ \varphi_{k,\ell}(x) = \frac{\cos \alpha_k(\ell) x}{\sqrt{1 + \frac{\sin 2\alpha_k(\ell)}{2\alpha_k(\ell)}}},\ \varphi_{k,\ell}^*(x) = \frac{\sin \alpha_k^*(\ell) x}{\sqrt{1 - \frac{\sin 2\alpha_k^*(\ell)}{2\alpha_k^*(\ell)}}} \tag{11}$$

for even and odd k, respectively. Therefore, the one-dimensional random field $q(x, \omega, \ell)$ with covariance function given by the equation (1) has the Karhunen-Loève expansion

$$q(x, \omega, \ell) = \lim_{M \to \infty} \sum_{k=1}^{M} \Big(\sqrt{c_{k,\ell}} \xi_k(\omega) \varphi_{k,\ell}(x) + \sqrt{c_{k,\ell}^*} \xi_k^*(\omega) \varphi_{k,\ell}^*(x) \Big). \tag{12}$$

For the truncated Karhunen-Loève expansion, one may take a $2M$-dimensional standard Gaussian space as probability space Ω, on which the random variables ξ_k and ξ_k^* are defined. It is clear that the solutions w and w^* of the deterministic equations (10) depend continuously on ℓ. The same is true for the eigenvalues as well as eigenfunctions shown in (11). Thus truncating the Karhunen-Loève expansion (12) at a finite M, the resulting random field depends continuously on the correlation length ℓ at fixed ω.

3.3. Random fields as random variables valued in function spaces

Consider the random field $q(x, \omega, \ell)$ as in Subsection 3.2. The *paths or trajectories (in space)* of the random field are the maps $x \to q(x, \omega, \ell)$, with fixed ω and ℓ. It is well-known from the theory of stochastic differential equations that the paths are continuous (Arnold, 1974). This also follows from the Kolmogorov-Chentsov theorem (Kallenberg, 1997), because the autocovariance function (1) satisfies the hypotheses of this theorem. However, for our purposes, more is needed.

First, jointly continuous dependence on x and ℓ is required. This is easy to achieve, because one can consider the *paths in space and parameter set* as the maps $(x, \ell) \to q(x, \omega, \ell)$, with fixed ω and apply the previous arguments. Indeed, when using a truncated Karhunen-Loève expansion, the argument was just given after (12). When the the Langevin equation is used, one may resort to the results of (Schmelzer, 2010; Schmelzer, 2013) which assert the continuous dependence, jointly in x and ℓ.

Second, the random fields have to be interpreted as random variables in the space of continuous functions. This is because they enter as variable coefficients in the partial differential equations to be

solved, and the solution depends on the whole trajectories, not only the values of the random fields at individual points x. Let x belong to an interval I and ℓ to an interval L, both closed and bounded. Continuity of the paths in x and ℓ can be stated by saying that the map $q(\omega) : (x, \ell) \to q(x, \omega, \ell)$ belongs to the space of continuous function $\mathcal{C}(I \times L)$ for fixed ω. Thus the random field induces a map $\Omega \to \mathcal{C}(I \times L)$, $\omega \to q(\omega)$. Standard arguments from the theory of stochastic processes show that this map is measurable, that is, a random variable with values in $\mathcal{C}(I \times L)$. For details of the proof, see (Nedeljković, 2020, Section 1.2). The argument also shows that

- for fixed $\ell \in L$, the map $\Omega \to \mathcal{C}(I)$, $\omega \to q(\omega, \ell)$ is measurable;
- for fixed $\omega \in \Omega$, the map $L \to \mathcal{C}(I)$, $\ell \to q(\omega, \ell)$ is continuous.

Note that the same random field q may be considered in three different ways as a measurable function of ω: as an element of the function space $\mathcal{C}(I \times L)$, denoted by $q(\omega)$, as an element of the function space $\mathcal{C}(I)$ at fixed correlation length ℓ, denoted by $q(\omega, \ell)$, and as a real number at each fixed x and ℓ, denoted by $q(x, \omega, \ell)$. Whenever its values are collected in a random set, the corresponding capital letter Q will be used.

3.4. The mean values of a parametrized random field as an interval field

Let $\omega \to A(\omega)$ be a random set whose focal elements $A(\omega) = [\underline{a}(\omega), \overline{a}(\omega)]$ are closed and bounded intervals. Let $\{a_k(\omega)$, k = 1,2,3,. . . $\}$ be a Castaing representation. The *Aumann expectation* $\mathrm{E}(A)$ is defined as the closure of the set of numbers $\{\mathrm{E}(a_k) : k = 1, 2, 3, \ldots\}$, see (Molchanov, 2005). It is clear that in the considered case, the Aumann expectation is simply the interval $\mathrm{E}(A) = [\mathrm{E}(\underline{a}), \mathrm{E}(\overline{a})]$. Next, consider a random set generated by a parametrized random field $\omega \to Q(x, \omega) = \{q(x, \omega, \ell) : \ell \in [\underline{\ell}, \overline{\ell}]\}$ as in Subsection 3.2. Thanks to the continuous dependence on the parameter ℓ discussed in Subsection 3.2 each $Q(x, \omega)$ is a closed and bounded interval at fixed x. One may form the Aumann expectation $\mathrm{E}(Q(x))$, which is a deterministic interval at fixed x. Thus the assignment $x \to \mathrm{E}(Q(x))$ defines an interval field.

4. Application to elastostatics

This section is devoted to a simple scalar partial differential equation with random field coefficients, in order to demonstrate how the random field methods can be applied in practice. The problem is

$$\begin{aligned} -\operatorname{div}(a(x)\operatorname{grad} u(x)) &= f(x), \quad x \text{ in } D \\ u(x) &= 0, \qquad x \text{ in } \partial D \end{aligned} \tag{13}$$

where D is a bounded open domain in $\mathbb{R}^d$ and ∂D is its boundary. For example, in two space dimensions u might be the transversal displacement of a non-uniform membrane under transversal load f. The spatially varying elastic properties are subsumed in the coefficient $a(x)$. Alternatively, in three space dimensions, $u(x)$ could be the pressure of a fluid in a porous medium with permeability $a(x)$ and source rate $f(x)$ (Matthies, 2008).

In the foundations of the finite element method, problem (13) is commonly solved by variational methods. Suitable function spaces are defined as follows: $L^2(D)$ is the space of square integrable functions; $H^1(D)$ is the space of square integrable functions all whose first order partial derivatives are square integrable as well; $H_0^1(D)$ is the subspace of functions that vanish on the boundary ∂D. Both $L^2(D)$ and $H_0^1(D)$ are normed spaces; the square of the norms are given by

$$\|f\|^2_{L^2(D)} = \int_D f(x)\,\mathrm{d}x, \quad \|u\|^2_{H^1(D)} = \|u\|^2_{L^2(D)} + \|\mathrm{grad\,u}\|^2_{L^2(D)}.$$

The spaces are complete and separable. The variational formulation of equation (13) then is:

Find $u \in H_0^1(D)$ such that

$$\int_D a(x)\,\mathrm{grad}\,u(x)\cdot\mathrm{grad}\,v(x)\,\mathrm{d}x + \int_D f(x)v(x)\,\mathrm{d}x = 0$$

for all $v \in H_0^1(D)$.

The deterministic case. Assume that a is a continuous function on D which is bounded from above and below, that is, $a \in \mathcal{C}(D)$ and $\alpha \le a(x) \le \beta$ for some constants $\alpha, \beta > 0$ and all x in D. Further, assume that f is square integrable on D. Under these conditions, it is well-known that the variational problem has a unique solution u in $H_0^1(D)$, (Nečas, 1967; Trèves, 1975). For the purpose of this example, the source term f will be fixed. The solution u depends on the coefficient function a; this dependence will be made explicit in the notation $u = u_a$. Denote the set of functions which are continuous on D up the boundary with values between α and β by $\mathcal{C}(D;\alpha,\beta)$. It is quipped with the norm $\|a\| = \max_{x\in D}|a(x)|$. The crucial assertion is that the map

$$\mathcal{C}(D;\alpha,\beta) \to H_0^1(D) : a \to u_a$$

is continuous. This non-trivial result can be proved by invoking the variational formulation, as it was done in (Nedeljković, 2020). A different proof, for more general partial differential equations, can be found in (Nečas, 1967).

The random field case. To start with, let $a(x,\omega)$, $\omega \in \Omega$, be a random field with continuous paths lying between α and β. As noted in Subsection 3.3, it can be viewed as a random variable with values in $\mathcal{C}(D;\alpha,\beta)$. Due to the continuity of the above map, the assignment $\omega \to u_{a(\omega)}$ is measurable, thus the solution u_a is a random variable with values in $H_0^1(D)$. Intuitively, this means that u_a is a random field whose paths belong to $H_0^1(D)$. In order to interpret it as a classical random field, one should be able to assign values $u_{a(\omega)}(x)$ at each point x in D. In one space dimension, this is easy because $H_0^1(D)$ is continuously imbedded in $\mathcal{C}(D)$, and so the pointwise evaluations

$$\mathcal{C}(D;\alpha,\beta) \to \mathbb{R} : a \to u_a(x)$$

are continuous as well. Thus the maps $\omega \to u_{a(\omega)}(x)$ are random variables for each x, and the paths $x \to u_{a(\omega)}(x)$ are continuous functions. In space dimensions two and three, more theory is needed. First, the input random field $a(x,\omega)$ should have Lipschitz continuous paths, that is, satisfying an inequality of the form $|a(x,\omega) - a(y,\omega)| \le C|x-y|$ for some constant C and all x, y. (This is not

true of the random field generated by the Langevin equation, but it is true of the random field generated by the truncated Karhunen-Loève expansion). Then one can invoke regularity theory saying that the solution to problem (13) is in $H^2(D')$ in any open subregion D' of D and depends continuously on a in this space (Nečas, 1967). Again, $H^2(D')$ is continuously imbedded in $\mathcal{C}(D')$, and so one obtains a classical random field $u_{a(\omega)}(x)$, defined for all x in the interior of D.

The random set case. Let $a(x)$ be a parametrized random field. More specifically, it will be constructed as follows. In two space dimensions, take a closed and bounded interval I such that $D \subset I \times I$ and a random field of the form $q(x, \omega, \ell)$ as in Subsection 3.2 and set

$$a(x, \omega, \ell) = \mu(x) + \chi(q(x_1, \omega, \ell)q(x_2, \omega, \ell)), \quad \text{for} \quad x = (x_1, x_2) \text{ in } D. \tag{14}$$

Here $\mu(x) > 0$ is the mean field (assumed fixed and deterministic) and χ is a cut-off function; this is needed because the individual realizations of a Gaussian random field can be arbitrarily large with non-zero probability. In the end, the cut-off has to be chosen so as to guarantee that $\alpha \leq a(x, \omega, \ell) \leq \beta$.

One could also use different correlation lengths in the two variables and/or multiply the fields with different standard deviations σ_1, σ_2. The construction in one or three space dimensions is analogous.

When ℓ varies in an interval L, a set-valued solution to (13) can be defined as $U(\omega) = \{u_{a(\omega,\ell)} : \ell \in L\}$. Combining the continuity of the map $a \to u_a$ with the statement at the end of Subsection 3.3, one obtains that

- for fixed $\ell \in L$, the map $\Omega \to H_0^1(D)$, $\omega \to u_{a(\omega,\ell)}$ is measurable;
- for fixed $\omega \in \Omega$, the map $L \to H_0^1(D)$, $\ell \to u_{a(\omega,\ell)}$ is continuous.

It follows from Subsection 3.1 that $U(\omega)$ is a random set in the target space $H_0^1(D)$. Again, in one space dimension one may form the set-valued functions $U(\omega, x) = \{u_{a(\omega,\ell)}(x) : \ell \in L\}$ at each x in D. By the same argument, the $U(\omega, x)$ are random sets in $\mathbb{R}$; they form the point evaluations of the random set solution $U(\omega)$ at the points x in D. In space dimensions two and three, a modified argument as in the random field case has to be applied.

Numerical example. The purpose of this subsection is to illustrate the method in a simple example, the displacement of an L-shaped membrane with non-uniform, random elastic properties. This has the character of a text book example, and so all units are taken dimensionless. The L-shaped domain D can be seen in Figure 2 (left). The equation for the displacement is (13). A constant load $f(x) \equiv 1$ is assumed, and the random field $a(x)$ is taken of the form (14) with constant mean $\mu(x) \equiv 1$, autocovariance function of the form (1), and field variance identically equal to one as well. In this elliptic problem, it suffices to guarantee that the realizations of the random field $a(x, \omega, \ell)$ stay above zero. Thus a lower cut-off function χ was chosen at the value 0.1. To produce the random set solution, the correlation length was taken from the interval $0.5 \leq \ell \leq 1.5$.

Numerically, the L-shaped region was subdivided in a grid of size 18×18, and problem (13) was solved by the finite element method using a Matlab-program. To generate the random field, the Karhunen-Loève expansion truncated at $M = 130$ was used. The value of $a(x, \omega, \ell)$ entered in each finite element was chosen as the average of the values at its four nodes. A Monte Carlo sample size

of 500 was used. At each realization of the underlying $2M$-dimensional Gaussian variable, see (12), the finite element program was executed with eleven values of ℓ between 0.5 and 1.5, stepsize 0.1.

As outlined above, this procedure results in an approximation to the random set solution $U(\omega)$. The solution is a random set in $H_0^1(D)$ and in $H^2(D')$ in any subregion, actually everywhere except near the sharp inside corner (Grisvard, 1985). Consequently, each horizontal or vertical slice through the solution is a random set in the space of continuous functions. At fixed $x = (x_1, x_2)$, each realization of the solution is an interval. The mean (Aumann expectation) is an interval field. Figure 2 (left) shows a realization of the displacement random field at fixed correlation length $\ell = 0.5$. Realizations of the full random set solution can be visualized by plotting slices at fixed x_1 or x_2. Such a cut at $x_2 = 0.4444$ is shown in Figure 2 (center). Above each fixed x_1, the realization is an interval. Finally, the mean value field can also be depicted by slices. Figure 2 (right) shows a slice of this interval field at $x_2 = 0.4444$. The upper and lower bounding curves of the mean displacement are shown together with the mean solutions at correlation lengths 0.5, 0.6, ..., 1.5.

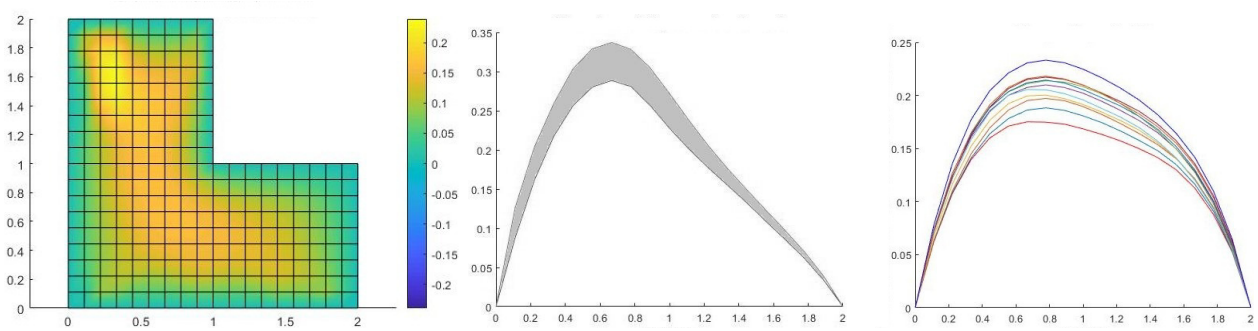

Figure 2. Some aspects of the random set solution. Trajectory of displacement random field at fixed correlation length $\ell = 0.5$ (left); slice at $x_2 = 0.4444$ through a realization of the full random set solution (center); slice at $x_2 = 0.4444$ through the mean value interval field, showing individual mean fields for varying correlation lengths inside.

5. Further results, numerical aspects

The parametric point of view. Given a parametrized family $a_\lambda, \lambda \in \Lambda$, of random variables or random fields, and a map $a_\lambda \to u_{a_\lambda}$ describing the response of a system, the approach favored in this paper has been to first construct the random set

$$U(\omega) = \{u_{a_\lambda(\omega)} : \lambda \in \Lambda\} \tag{15}$$

and then to define lower and upper probabilities of events B by

$$\underline{P}(B) = P(U_-(B)), \quad \overline{P}(B) = P(U^-(B)).$$

An alternative approach, the *parametric viewpoint*, would consist in skipping the random set step and defining lower and upper probabilities induced by the family $a_\lambda, \lambda \in \Lambda$, directly:

$$P_{\text{low}}(B) = \inf_{\lambda\in\Lambda} P(u_{a_\lambda}), \quad P_{\text{upp}}(B) = \sup_{\lambda\in\Lambda} P(u_{a_\lambda}).$$

For the events $B = (-\infty, b]$ one obtains again upper and lower distribution functions and a p-box as in Figure 1, which contains, however, only the cumulative distribution functions of the underlying family (and hence is often referred to as a *parametric p-box*). This approach gives tighter bounds (Fetz and Oberguggenberger, 2016):

$$\underline{P}(B) \leq P_{\text{low}}(B) \leq P_{\text{upp}}(B) \leq \overline{P}(B)$$

but deprives one of using the concepts of the theory of random sets. The numerical computation of the two types of upper and lower probabilities are different, but of comparable effort.

Numerical aspects. As is well-known, the computational effort when implementing heterogeneous uncertainty models is high: it involves both Monte Carlo sampling and the computation of interval bounds. Suppose the quantity of interest, given through a model response function $a_\lambda \to u_{a_\lambda}$ as above, is one-dimensional. (This could be the model response at a single point x or the value of a performance criterion.) Further, suppose the lower and upper distribution functions $\underline{F}(b)$, $\overline{F}(b)$ of this quantity are sought.

In both approaches, the parameter set Λ has to be discretized in a grid $\lambda_1, \ldots, \lambda_M$. In the random set case, one first produces a Monte Carlo sample $\omega_1, \ldots, \omega_N$ on the underlying probability space, then computes the bounding functions $\underline{u}(\omega_k)$, $\overline{u}(\omega_k)$ of the sets formed by the u_{a_λ} over all λ_i at fixed ω_k and evaluates the sample of bounding functions statistically. In the parametric point of view, one starts with a Monte Carlo sample of size N of each random variable a_{λ_i}, propagates it through the model function u, evaluates the result statistically at each λ_i and then finds the upper lower bounds of the resulting probabilities. A more detailed exposition of the two procedures can be found in (Fetz and Oberguggenberger, 2016).

In these crude forms, both algorithms require $M \cdot N$ evaluations of the (generally expensive) model function u. Accordingly, many approaches have been devised to reduce computational cost. In the random set case, a stochastic response surface, like a polynomial chaos expansion (Ghanem and Spanos, 1991), reduces the number M to a level required for the accuracy of the response surface. Evaluating a Monte Carlo sample of size N, given the response surface, is cheap (Oberguggenberger, 2015). For further sampling concepts for random sets, see (Alvarez, 2006). Other methods include approximations by inner and outer bounds (Alvarez, 2009; Alvarez, Hurtado and Ramírez, 2017; Tonon, 2004). In the parametric point of view, a prominent method is reweighting a single importance Monte Carlo sample, which means $M = 1$. Still N model evaluations are required (Fetz, 2019; Troffaes, 2018; Zhang and Shields, 2018). (This can of course be aided by a deterministic response surface.) Very efficient methods appear to be *advanced line sampling* (de Angelis, Patelli and Beer, 2015) that intertwines the two required loops and can reduce M and N simultaneously, and the method of *probability plots*, an enhancement of the first order reliability method (FORM) proposed by (Hurtado, Alvarez and Paredes, 2017).

6. Conclusion

The paper addressed mathematical and modelling issues when heterogeneous uncertainty is entered in engineering models. It was demonstrated that the framework of random sets is suitable for this task, both from the theoretical and computational viewpoint. It was shown how the theoretical

measurability questions can be addressed, and that the continuous dependence of the solution and the involved random and interval quantities on their parameters is important. This gives a solid basis for the commonly employed numerical methods. As was shown in (Nedeljković, 2020), the methods can also be applied to dynamic problem, such as wave or transport equations.

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