Why Ellipsoids in Mechanical Analysis of Wood Structures

F. Niklas Schietzold[1], Julio Urenda[2,3], Vladik Kreinovich[3], Wolfgang Graf[1], and Michael Kaliske[1]

[1] *Institute for Structural Analysis, Technische Universität Dresden*
01062 Dresden, Germany, {niklas.schietzold, wolfgang.graf, michael.kaliske}@tu-dresden.de

[2,3] *Departments of* [2] *Mathematical Sciences and* [3] *Computer Science*
University of Texas at El Paso, El Paso, TX 79968, USA, {jcurenda, vladik}@utep.edu

Abstract. Wood is a very mechanically anisotropic material. At each point on the wooden beam, both average values and fluctuations of the local mechanical properties corresponding to a certain direction depend, e.g., on whether this direction is longitudinal, radial or tangential with respect to the grain orientation of the original tree. This anisotropy can be described in geometric terms, if we select a point x and form *iso-correlation* surfaces – i.e., surfaces formed by points y with the same level of correlation $\rho(x, y)$ between local changes in the vicinities of the points x and y. Empirical analysis shows that for each point x, the corresponding surfaces are well approximated by concentric homothetic ellipsoids. In this paper, we provide a theoretical explanation for this empirical fact.

Keywords: wood mechanical properties, ellipsoids, anisotropy

1. Formulation of the Problem: Need for a Theoretical Explanation of an Empirical Fact

Many constructions are made of wood. Wood is one of the oldest materials used in construction. During the past millennia, people have developed a lot of skills for working with wood. However, in spite of this experience, wood remains one of the most difficult materials to handle. The main reason for this difficulty is that, in contrast to many other construction materials which are mostly homogeneous and isotropic, wood is highly inhomogeneous and anisotropic. At each point in the wooden beam, both the average values and fluctuations of the local mechanical properties corresponding to a certain direction depend, e.g., on whether this direction is longitudinal, radial or tangential with respect to the grain orientation of the original tree. In designing wooden constructions, it is important to properly describe and to properly take into account this inhomogeneity and anisotropy; see, e.g., (Vorreiter, 1949; Kollman, 1982; Reuter, 2009; Ulrich and Seim, 2014; Jenkel et al., 2015; Leichsenring et al., 2018; Schietzold, Graf, and Kaliske, 2018; Fleischhauer et al., 2019).

How can we describe local fluctuations of mechanical characteristics? These fluctuations are caused by many different relatively small factors. It is known that the distribution of the joint effect of a large number of relatively small factors is close to Gaussian – this follows from the Central Limit Theorem, according to which this distribution tends to Gaussian when the number of factors increases; see, e.g., (Sheskin, 2011). To describe a Gaussian distribution, it is sufficient

to describe its first and second moments. For a general random field $f(x)$, this means that we need to describe its mean values $E[f(x)]$ (where $E[\cdot]$ denotes the expected value) and its covariances $E[f(x)\cdot f(y)]$. For fluctuations, the mean is 0, so we only need to describe covariances. In statistics, it is often convenient, instead of explicitly describing covariances, to describe the standard deviations $\sigma[f(x)] \stackrel{\text{def}}{=} \sqrt{E[(f(x)^2]}$ and correlations $\rho(x,y) \stackrel{\text{def}}{=} \dfrac{E[f(x)\cdot f(y)]}{\sigma[f(x)]\cdot\sigma[f(y)]}$. Then, covariances can be reconstructed as $E[f(x)\cdot f(y)] = \sigma[f(x)]\cdot\sigma[f(y)]\cdot\rho(x,y)$.

An interesting property of the corresponding correlation functions was recently empirically found; see, e.g., (Schietzold, Graf, and Kaliske, 2019) and references therin. This property is about *iso-correlation* surfaces corresponding to each spatial location x, i.e., surfaces formed by all the points y for which the correlation $\rho(x,y)$ is equal to a constant ρ_0. Empirical analysis shows that for each point x, the corresponding surfaces are well approximated by concentric homothetic ellipsoids; see, e.g., (Schietzold, Graf, and Kaliske, 2019). This property helps narrow down possible functions $\rho(x,y)$ when we analyze mechanical properties of wood – and thus, has a potential to make mechanical analysis of wooden structures more efficient.

The problem is that so far, this property was purely empirical, it had no theoretical justification, and thus, engineers were reluctant to use it: it is known that sometimes, empirical properties found under some conditions do not work well when conditions change. To make this property more reliable and thus, more practically useful, it is therefore desirable to come up with a theoretical explanation.

In this paper, we provide a desired theoretical explanation for this empirical fact.

2. Our Explanation: Main Idea

The main ideas behind our explanation are similar to the ideas used in (Finkelstein, Kosheleva, and Kreinovich, 1996; Li, Ogura, and Kreinovich, 2002) to explain efficiency of ellipsoid approximation in numerical analysis (see, e.g., (Schweppe, 1968; Schweppe, 1973; Fogel and Huang, 1982; Belforte and Bona, 1985; Norton, 1985; Chernousko, 1988; Soltanov, 1990; Utyubaev, 1990; Filippov, 1992; Chernousko, 1994)); the main difference is that now we consider:

- *not* classes of sets (such as the class of all ellipsoids), but
- classes of *families* of sets (e.g., the class of all families of concentric homothetic ellipsoids).

Specifically, we show that for the smallest dimension d for which it is possible to have an affine-invariant optimality criterion on the space of all such d-dimensional classes, for any such criterion, the optimal family consists of concentric homothetic ellipsoids. Thus, such families of ellipsoids provide the optimal approximation to the actual surfaces – at least in the *first* approximation, i.e., approximation corresponding to the smallest possible number of parameters.

3. Our Explanation: Details

Family of sets: towards a precise definition. For each spatial point x, we would like to describe, for each possible value ρ_0 of the correlation $\rho(x, y)$, the set $S_{\rho_0}(x)$ of all the points y for which the correlation $\rho(x, y)$ between the values at x and y is greater than or equal to ρ_0.

What are the natural properties of these families of sets?

First property: coverage. For each y, there is some value of $\rho(x, y)$, so for this x, the union of all these sets $S_{\rho_0}(x)$ coincides with the whole space.

Second property: monotonicity. Of course, if $\rho(x, y) \geq \rho_0$ and $\rho_0 \geq \rho_0'$, then $\rho(x, y) \geq \rho_0'$. So, the sets $S_{\rho_0}(x)$ should be inclusion-monotonic: if $\rho_0 \leq \rho_0'$, then $S_{\rho_0'}(x) \subseteq S_{\rho_0}(x)$.

Third property: boundedness. From the physical viewpoint, the further away is the point y from the point x, the less the physical quantities corresponding to these points are correlated. As the distance increases, this correlation should tend to 0. Thus, each set $S_{\rho_0}(x)$ is *bounded*.

Fourth property: continuity. In physics, most processes are continuous – with the exception of processes like fracturing, which we do not consider here. We can therefore conclude that the correlation $\rho(x, y)$ continuously depends on y. So, if we have $\rho(x, y_n) \geq \rho_0$ for some sequence of points y_n that converges to a point y ($y_n \to y$), then we should have $\rho(x, y) = \lim\limits_{n\to\infty} \rho(x, y_n) \geq \rho_0$. In other words, if $y_n \in S_{\rho_0}(x)$ and $y_n \to y$, then $y \in S_{\rho_0}(x)$, i.e., each set $S_{\rho_0}(x)$ is *closed*.

Similarly, it is reasonable to conclude that the set $S_{\rho_0}(x)$ should continually depend on ρ_0, i.e., that of the two values ρ_0 and ρ_0' are close, then the corresponding sets $S_{\rho_0}(x)$ and $S_{\rho_0'}(x)$ should also be close. A natural way to describe closeness between (bounded closed) sets is to use the so-called Hausdorff distance. In precise terms, for any $\varepsilon > 0$, we say that the sets A and B are *ε-close* if:

- every point $a \in A$ is ε-close to some point $b \in B$ (in the sense that the distance $d(a, b)$ does not exceed ε: $d(a, b) \leq \varepsilon$), and
- every point $b \in B$ is ε-close to some point $a \in A$.

The Hausdorff distance $d_H(A, B)$ between the sets A and B is then defined as the smallest ε for which the sets A and B are ε-closed. It can be shown that this distance can be equivalently defined as follows:

$$d_H(A, B) = \max\left(\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right),$$

where $d(a, B) \stackrel{\text{def}}{=} \inf\limits_{b\in B} d(a, b)$.

Fifth property: what is the set of possible values of the parameter? In this family of sets, correlation value is a parameter. What are the possible values of correlation? In general, correlations can take any value from -1 to 1. When $y = x$, the correlation is clearly equal to 1. When $y \to \infty$, we get values close to 0. Since the function $\rho(x, y)$ is continuous, this function takes all intermediate values. So, the possible values of the correlation form some interval. In some cases, we may have all possible negative values, in other cases, only some negative values, in yet other cases, we only

have non-negative values. So, in general, we will consider all possible intervals of possible value of ρ_0. This interval may be closed – if there are points with - correlation, or is can be open.

Resulting definition. So, we arrive at the following definition.

Definition 1. *Let $N \geq 2$ be an integer. By a* family of sets*, we mean a set $\{S_c : c \in I\}$ of bounded closed sets $S_c \subseteq \mathbb{IR}^N$ obtained by applying, to each real number c from a non-degenerate interval I (open or closed, finite or infinite), a mapping $c \to S_c$ that has the following properties:*

- *the dependence of S_c on c is continuous: if $c_n \to c$, then $d_H(S_{c_n}, S_c) \to 0$,*
- *the family S_c is monotonic: if $c < c'$, then $S_{c'} \subseteq S_c$, and*
- *the union of all the sets S_c coincides with the whole space.*

Comments.

- According to this definition, the family remains the same if we simply re-parameterize the family: e.g., if instead of the original parameter c, we use a new parameter $c' = c + c_0$ or $c' = \lambda \cdot c$ for some constants c_0 and λ.
- In our specific problem, we are interested in the 3-D case $N = 3$. However, since we can envision similar problem in the plane $N = 2$ or in higher-dimensional spaces – and since the proof of our main result does not depend on any specific value N – in this paper, we consider the general case $N \geq 2$.
- We are specifically interested in *concentric homothetic families of ellipsoids*, i.e., in families of the type $S_c = c \cdot E + a$, where a is a given vector, and E is an ellipsoid with a center at 0.

Class of families of sets. For different situation, in general, we get different correlations and thus, different families of sets. We would like to find general class of such families that would, ideally, cover all such situations. We can use different parameters to differentiate different families from this class. In other words, a class can be described as a method for assigning, to each possible combination of values of these parameters, a specific family. As before, it makes sense to require that the resulting mapping is continuous.

Definition 2. *Let $N \geq 2$ and $r > 0$ be integers. By an r-*parametric class of families of sets*, we mean a mapping that assigns, to each element $p = (p_1, \ldots, p_r)$ from an open r-dimensional set $D \subseteq \mathbb{IR}^r$, a family $\{S_c(p)\}$ so that the dependence of $S_c(p)$ on c and p is continuous.*

Optimality criteria: general idea. Out of all possible classes, we want to select a class which is, in some reasonable sense, optimal. For this, we need to be able to describe when some classes are better than others. In other words, we need to have an *order* on the set of all the classes. It would be nice to have a *total* (*linear*) order, in the sense that for every two classes, we should be able to tell which one is better, but it may be sufficient to have a *partial* order – as long as this order enables us to select the best class, it is OK if some not-best classes, we do not have an opinion of which of them is better.

In practice, usually, optimality criteria are described in numerical form: we have an objective function $f(a)$ that assigns a numerical value to each possible alternative a, and we want to select an alternative for which this value is the largest possible (or, depending on the context, the smallest possible). For example, a company wants to maximize its profit, a city wants to upgrade its road system so as to minimize the average travel time, etc.

However, often, we need to go somewhat beyond this approach. Indeed, for example, a company may have two (or more) different projects that lead to the same expected profit. In this case, we can use this non-uniqueness to optimize something else – e.g., out of all most-profitable projects, select the one that leads to the smallest possible long-term environmental impact. In this case, we have a more complex criterion for comparing alternatives: instead of saying that an alternative a is better than the alternative a' if $f(a) > f(a')$, we say that a is better if:

- either $f(a) > f(a')$
- or $f(a) = f(a')$ and $g(a) > g(a')$, for some other numerical criterion $g(a)$.

If this still does not select us a unique alternative, we can optimize yet something else, etc. In view of this possibility, in this paper, we do not restrict ourselves to numerical optimization criteria and use the most general definition of the optimality criterion, when:

- for some pairs of alternatives a and a', we know that a is better (we will denote it by $a' < a$),
- for some pairs of alternatives a and a', we know that a' is better ($a < a'$), and
- for some pairs of alternatives a and a', a and a' are of the same value (we will denote it by $a \sim a'$).

Clearly, if a' is better than a, and a'' is better than a', then a'' should be better than a, etc. Thus, we arrive at the following definition:

Definition 3. *Let A be a set; elements of this set will be called* alternatives*. By an* optimality criterion*, we mean a pair of binary relations $(<, \sim)$ on the set A for which the following properties hold:*

- *if $a < a'$ and $a' < a''$, then $a < a''$;*
- *if $a < a'$ and $a' \sim a''$, then $a < a''$;*
- *if $a \sim a'$ and $a' < a''$, then $a < a''$;*
- *if $a \sim a'$ and $a' \sim a''$, then $a \sim a''$;*
- *if $a \sim a'$, then $a' \sim a$;*
- *if $a < a'$, then we cannot have $a' < a$ or $a \sim a'$.*

Comment. Such a pair of relations is sometimes called a *partial pre-order*.

Definition 4. *Let $(<, \sim)$ be an optimality criterion on a set A. An alternative a_{opt} is called* optimal *with repect to this criterion if for every alternative $a \in A$, we have $a < a_{\text{opt}}$ or $a \sim a_{\text{opt}}$.*

We need a final optimality criterion. If an optimality criterion does not select any alternative as optimal, this means that this criterion still needs work – this may happen if for most pairs of alternatives, this criterion does not tell us which alternative is better. So, for the optimality criterion to be useful, it must select *at least one* optimal alternative.

If the criterion selects *several* alternatives as optimal, this means – as we have mentioned earlier – that this criterion is not final: we can use the resulting non-uniqueness to optimize something else, i.e., in effect, to come up with a better optimality criterion. If for this better criterion, we still have several optimal alternatives, we can (and should) modify this criterion even further, etc., until we finally get a criterion for which there is exactly one optimal alternative. Thus, we arrive at the following natural definition.

Definition 5. *We say that an optimality criterion is* final *if there exists exactly one alternative which is optimal with respect to this criterion.*

For our problem, an optimality criterion must be affine-invariant. In our case, we want to compare different classes (of families of sets). In selecting optimality criteria, it is reasonable to take into account that while we want to deal with sets of points in physical space, from the mathematical viewpoint, we deal with sets of tuples of real numbers. Real numbers (coordinates) describing each point depend on what coordinate system we use: if we select a different starting point, then all the coordinates are shifted $x_i \to x_i + a_i$; if we select different axes for the coordinates, we get a rotation $x_i \to \sum\limits_{j=1}^{N} r_{ij} \cdot x_j$ for an appropriate matrix r_{ij}, etc.

These transformations make sense for the *isotropic* case, when all the properties of a material are the same in all directions. Wood is an example of an *anisotropic* material: e.g., it is easier to cut it along the orientation of the original tree than across that orientation. It is known that in many cases, the description of an anisotropic material can be reduced to the isotropic case if we apply an appropriate affine transformation. This usually comes from the fact that, e.g., mechanical properties of a body can be described by a symmetric matrix, and each symmetric matrix can become a unit matrix if we use its eigenvectors as the base for the new coordinate system.

In view of this, it is reasonable to require that our optimality criterion is invariant not only with respect to shifts and rotations, but also with respect to all possible affine (linear) transformations. Thus, we arrive at the following definitions.

Definition 6. *Let $N > 2$ be an integer. By an* affine transformation*, we mean a transformation $T : \mathbb{IR}^N \to \mathbb{IR}^N$ of the type $(Tx)_i = a_i + \sum\limits_{j=1}^{N} b_{ij} \cdot x_j$ for some reversible matrix b_{ij}. Let T be an affine transformation.*

- *Let $S \subseteq \mathbb{IR}^N$ be a set. By the* result $T(S)$ *of applying T to S, we mean the set $\{T(s) : s \in S\}$.*
- *Let $F = \{S_c : c \in I\}$ be a family of sets. By the* result $T(F)$ *of applying T to F, we mean the family $\{T(S_c) : c \in I\}$.*

– *Let $C = \{S_c(p)\}$ be class of families. By the* result *$T(C)$ of applying T to C, we mean the class $\{T(S_c(p))\}$.*

Definition 7. *Let A be a set of alternatives, let $(<, \sim)$ be an optimality criterion of the set A, and let $\mathcal{T}$ be a class of transformations $A \to A$. We say that the optimality criterion $(<, \sim)$ is* $\mathcal{T}$-invariant *if for every $T \in \mathcal{T}$ and for all $a, a' \in A$, the following two properties hold:*

– *if $a < a'$ then $T(a) < T(a')$, and*

– *If $a \sim a'$, then $T(a) \sim T(a)$.*

Proposition 1. *Let $N > 0$ and $r > 0$ an integers, and let $(<, \sim)$ be a final affine-invariant optimality criterion on the set of all r-parametric classes of families of sets in $\mathbb{R}^N$. Then:*

– $r \geq \dfrac{N \cdot (N+3)}{2} - 1$*; and*

– *for $r = \dfrac{N \cdot (N+3)}{2} - 1$, the optimal class consists of concentric homothetic families of ellipsoids.*

Comment. This result indeed shows that the class of concentric homothetic families of ellipsoids is the simplest of all possible optimal classes – simplest in the sense that it requires the smallest number of parameters to describe.

4. Proof

1°. Since the optimality criterion is final, there exists exactly one class C_{opt} which is optimal with respect to this criterion, i.e., for which $C < C_{\text{opt}}$ or $C \sim C_{\text{opt}}$ for all other classes C. Let us first prove that the optimal class C_{opt} is itself affine-invariant, i.e., that $T(C_{\text{opt}}) = C_{\text{opt}}$ for each affine transformation T.

Indeed, due to optimality, for each class C and for each affine transformation class T, for the class $T^{-1}(C)$, i.e., we have either $T^{-1}(C) < C_{\text{opt}}$ or $T^{-1}(C) \sim C_{\text{opt}}$.

Since the criterion is affine-invariant, we have either $T(T^{-1}(C)) < T(C_{\text{opt}})$ or $T(T^{-1}(C)) \sim T(C_{\text{opt}})$. Here, by the definition of the inverse transformation, $T(T^{-1}(C)) = C$, so we conclude that for every class C, we have either $C < T(C_{\text{opt}})$ or $C \sim T(C_{\text{opt}})$. By definition of optimality, this means that the class $T(C_{\text{opt}})$ is optimal. However, our optimality criterion is final, which means that there is only one optimal class. Thus, indeed, $T(C_{\text{opt}}) = C_{\text{opt}}$.

Since the optimal class is affine-invariant, with each family F this class also contains the family $T(F)$. This means, in its turn, that for each set S_c from each family, some family from the optimal class also contains the set $T(S_c)$.

2°. Let us show that $r \geq \dfrac{N \cdot (N+3)}{2} - 1$. Indeed, it is known (see, e.g., (Busemann, 1955)) that for every non-degenerate bounded set S (i.e., for every bounded set which is not contained in a

proper subspace), among all ellipsoids that contain S, there exists a unique ellipsoid of the smallest volume. It is also known that this correspondence between a set and the corresponding ellipsoid is affine-invariant: if an ellipsoid E corresponds to the set S_c, then, for each affine transformation T, to the set $T(S_c)$ there corresponds the ellipsoid $T(E)$.

It is known that every two ellipsoids can be obtained from each other by an appropriate affine transformation. Thus, the family of all ellipsoids corresponding to all the sets from all the families consists of all the ellipsoids. How many ellipsoids are there? A general ellipsoid can be determine by a quadratic formula $\sum\limits_{ij} a_{ij} \cdot x_i \cdot x_j + \sum\limits_{i=1}^{N} a_i \cdot x_i \le 1$ for some symmetric matrix a_{ij} and a vector a_i – and it is easy to see that different combinations of the matrix and the vector lead to different ellipsoids. We need N values $a_1, \ldots, a_N$ to describe a vector. Out of N^2 elements of the matrix, we need N values to describe its diagonal values a_{ii} and we need $\dfrac{N^2-N}{2}$ to describe non-diagonal elements: we divide by two since the matrix is symmetric $a_{ij} = a_{ji}$. Thus, overall, we need

$$N + N + \frac{N^2-N}{2} = \frac{N \cdot (N+3)}{2}$$

values.

Thus, the set of all ellipsoids is $\dfrac{N \cdot (N+3)}{2}$-dimensional. Since to each set S_c from families from the optimal class, we assign an ellipsoid, the dimension of the set of such sets should also be at least $\dfrac{N \cdot (N+3)}{2}$-dimensional. These sets are divided into 1-parametric families, so the dimension r of the class of such families cannot be smaller than the above dimension minus 1. Thus, indeed,

$$r \ge \frac{N \cdot (N+3)}{2} - 1.$$

3°. Let us now prove that for the smallest possible dimension $r = r_{\min} \stackrel{\text{def}}{=} \dfrac{N \cdot (N+3)}{2} - 1$, all the sets S_c from the each family of the optimal class are ellipsoids.

In Part 2 of this proof, we showed that each ellipsoid is associated with some set S_c from one of these families. The unit ball with a center at 0 is clearly an ellipsoid. Let us consider the set S_c which is associated with this unit ball. A unit ball is invariant with respect to all the rotations around its center. If the associated set S_c is not equal to the unit ball, this means that this set is not invariant with respect to at least some rotations. In other words, the group of all rotations that leave this set invariant is a proper subgroup of the group of all rotations. This implies that the dimension of this group is smaller than the dimension of the group of all rotations – and thus, that there exists at least 1-parametric family $\mathcal{R}$ of rotations R with respect to which the set S_c is not invariant.

Since the optimal class is affine-invariant, all the sets $R(S_c)$ are also sets from some family from the optimal class – and for all of them, the same unit ball is the smallest-volume ellipsoid. Thus, for this particular ellipsoid – the unit ball, we have at least a 1-dimensional family of sets S_c associated with this same ellipsoid. By applying a generic affine transformation, we can find a

similar at-least-1-dimensional family of sets corresponding to each ellipsoid. Thus, the dimension of the set of all sets S_c is at least one larger than the dimension of the family of all ellipsoids, i.e. at least $\frac{N \cdot (N+3)}{2} + 1 = r_{\min} + 2$. However, we have a $r_{\min}$-dimensional class of 1-dimensional families of sets, so the overall dimension of the set of all the sets S_c cannot be larger than $r_{\min} + 1$. This contradiction shows that the set S_c cannot be different from the enclosing minimal-volume ellipsoid. Thus, indeed, each set from each family from the optimal class is an ellipsoid.

4°. To complete the proof, we need to prove that ellipsoids in each family are concentric and homothetic.

We have proven that each ellipsoid appears as an appropriate smallest-volume set. Now that we know that each set S_c coincides with its smallest-volume enclosure, we can thus conclude that each ellipsoid appears as one of the sets S_c from one of the families from the optimal class. Similarly to Part 3 of this proof, let us consider the unit ball centered at 0. If the 1-dimensional family F_0 containing this ball is not invariant with respect to all possible rotations around the ball's center, then we have at least a 1-dimensional group of different families containing the same ellipsoid – the unit ball. However, the only way for an $r_{\min}$-dimensional class of 1-dimensional families to cover the whole $(r_{\max} + 1)$-dimensional family of ellipsoids is when all elements of all families are different. So we cannot have several families containing the same ellipsoid.

This argument shows that the family F_0 containing the unit ball *should be* rotation-invariant. Since all the sets from this family are included in each other and thus, cannot be transformed into each other by rotations – this means that each ellipsoid from this family F_0 must be rotation-invariant. This means that each ellipsoid from this family must be a ball concentric with our selected unit ball – and thus, homothetic to this ball.

For any other family F, by selecting any ellipsoid E from this family and applying the affine transformation that transforms the above unit ball into E, we get a new family $T(F_0)$ of concentric homothetic ellipsoids. Since an ellipsoid can only belong to one family, we thus conclude that the family F also consists of concentric homothetic ellipsoids.

The proposition is proven.

5. Conclusions

Wood is one the oldest construction materials. However, in spite of several thousand years of experience with wooden constructions, predicting and estimating mechanical properties of wooden constructions remains a difficult problem. One of the main reasons for this difficulty is that, in contrast to many other constructions materials which are largely homogeneous and isotropic, wood is highly inhomogeneous and anisotropic. Recently, a new property of wooden materials was discovered that has a potential to make mechanical analysis of wooden structures more efficient: that for wood, iso-correlation surfaces (i.e., surfaces of equal correlation) are well-approximated by concentric homothetic ellipsoids. The problem is that this property is purely empirical, it has no theoretical explanation and thus, engineers are understandably reluctant to rely on it in their estimates. In this papet, we provide a theoretical explanation for this empirical fact and thus, make this property more reliable and therefore more useable.

Acknowledgements

This work was supported in part by the US National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science) and HRD-1242122 (Cyber-ShARE Center of Excellence).

The authors are thankful to the anonymous referees for valuable suggestions.

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