Explicit sensitivities of the stochastic response of structural systems under spectrum compatible fully non-stationary seismic excitations

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Abstract: The sensitivity analysis is a suitable vehicle to evaluate the variation of structural responses under the influence of changes of structural parameters. It has been recently recognized that ground motion accelerations change both in their amplitude and frequency content and can be classified as non-stationary processes. It follows that the sensitivity of the response *evolutionary power spectral density* (*EPSD*) of structures subjected to non-stationary stochastic processes is an essential information and, consequently, it plays a fundamental role in structural design.
In this study handy expressions for the evaluation of sensitivities of stochastic response characteristics of structural systems with damping devices subjected to seismic excitations, modeled as fully non-stationary Gaussian stochastic spectrum compatible processes, are evaluated. Since the structural systems with damping devices are non-classically damped, first, the *time-frequency varying response* (*TFR*) function vector for non-classically damped systems is obtained in explicit form. Then, closed form solutions for the sensitivities of the *TFR*, as well as of the one-sided *EPSD* of the structural response, with respect to device parameters are evaluated.

Keywords: Sensitivity analysis, Complex modal analysis, Stochastic analysis, Fully non-stationary processes.

1. Introduction

The sensitivity analysis (i.e. the evaluation of partial derivatives of a performance measure with respect to system parameters) is a suitable vehicle to evaluate the variation of structural responses under the influence of changes of structural parameters (Arora and Haug, 1979). It follows that the sensitivity analysis plays a fundamental role in structural design.

Strong motion earthquakes are certainly the main critical actions for structures located in the seismically active regions of the earth. The analysis of recorded accelerograms after earthquakes evidence that different earthquakes produce ground motions with different characteristics; that is, ground motions with different intensity, duration, dominant periods and frequency content. It follows that in order to guaranty a good performance of structures in seismic areas a correct characterization of the ground motion acceleration is necessary. The analysis of recorded accelerograms in different sites shows that earthquake ground motion time-histories can be considered as sample of a zero mean Gaussian non-stationary processes in both amplitude and frequency content: the so-called *fully non-stationary processes*. It follows that the structural response is a stochastic process which must be coherently evaluated by applying the tools of stochastic dynamics. It is fully characterized by the so-called *evolutionary power spectral density* (*EPSD*) function matrix.

The stochastic sensitivity, that is the variation of statistics of structural stochastic responses as a consequence of structural parameters modifications, was pioneering studied by Szopa (1984) and Socha (1986). Afterward several papers have been devoted to this subjected. As an example, Benfratello et al. (2000) proposed a procedure, in the time domain, to evaluate the sensitivity of the statistical moments of the response for stationary Gaussian and non-Gaussian white input processes. Chaudhuri and Chakraborty (2004) developed the formulation in double frequency domain to tackle non-stationary excitation for obtaining the analytical sensitivity statistics of various dynamic response quantities with respect to structural parameters. Cacciola et al. (2005) proposed a numerical procedure for the determination of the evolution of the response statistics sensitivity for both classically and non-classically damped structural systems subjected to non-stationary non-white input processes, by solving set of differential equations once the Kronecker algebra was applied. Liu (2012) proposed numerical methods for the calculation of the sensitivity and Hessian matrix of the response *power spectral density* (*PSD*) matrix function of structures subjected to uniformly modulated evolutionary random seismic excitation. The methods were formulated by accompanying the pseudo-excitation method (Lin et al, 1994) with the Gauss precise time step method or the Newmark method, respectively. Yan et al (2017), utilizing the pseudo-excitation method, implemented a procedure to evaluate the sensitivity of first and second order of response *PSD* functions once the derivatives of eigenpair are evaluated.

In this study handy expressions for the evaluation of sensitivities of stochastic response characteristics of structural systems with damping devices subjected to seismic excitations, modelled as fully non-stationary Gaussian stochastic spectrum compatible processes, are evaluated. Since the structural systems with damping devices are non-classically damped, first, according to the formulation recently proposed by Alderucci and Muscolino (2018), the *time-frequency varying response* (*TFR*) function for non-classically damped systems is evaluated in explicit form. Then, closed form solutions for the first-order derivatives of the *TFR* as well as of the one-sided *EPSD* of the structural response, with respect to device parameters are evaluated.

Numerical applications show the computational efficiency of the proposed approach.

2. Problem Formulation

2.1 Equations of motion

Let us consider a structural system subjected to seismic excitations whose configuration could be modified for design reasons introducing seismic devices. It follows that the equations of motion of an *n*-degree of freedom (*n*-DOF) structural linear system, quiescent at time $t=0$, can be written in the form

$$\mathbf{M}\ddot{\mathbf{U}}(t,\boldsymbol{\alpha})+\mathbf{C}(\boldsymbol{\alpha}_C)\dot{\mathbf{U}}(t,\boldsymbol{\alpha})+\mathbf{K}(\boldsymbol{\alpha}_K)\mathbf{U}(t,\boldsymbol{\alpha})=-\mathbf{M}\boldsymbol{\tau}\ddot{U}_{\mathrm{g}}(t);\quad \mathbf{U}(t_0,\boldsymbol{\alpha})=\mathbf{0}\,, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}(\boldsymbol{\alpha}_C)$, and $\mathbf{K}(\boldsymbol{\alpha}_K)$ are the $n\times n$ mass, damping, and stiffness matrices of the structure, $\mathbf{U}(\boldsymbol{\alpha},t)$ is the *n*-dimensional vector of nodal displacements relative to the ground; τ is the *n*-dimensional array listing the influence coefficients of the ground shaking; $\ddot{U}_{\mathrm{g}}(t)$ is the seismic acceleration; a dot over a variable denotes differentiation with respect to time. In Eq.(1), the dependence of the damping and stiffness matrices of the structure, as well as of the response vector, on the unknown *r*-order parameter vector α, characterizing seismic devices, is stressed. The vector $\boldsymbol{\alpha}^T = \left[\boldsymbol{\alpha}_C^T \quad \boldsymbol{\alpha}_K^T\right]$, of order $r\ (r = r_c + r_k)$, collects seismic device parameters, which must be evaluated by the design procedure. It can be split as:

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \Delta\boldsymbol{\alpha}\,, \tag{2}$$

where $\Delta\boldsymbol{\alpha}^T = \left[\Delta\boldsymbol{\alpha}_C^T \quad \Delta\boldsymbol{\alpha}_K^T\right]$ is assumed to be a vector collecting small parameter variations with respect to the nominal parameter vector $\boldsymbol{\alpha}_0^T = \left[\boldsymbol{\alpha}_{C,0}^T \quad \boldsymbol{\alpha}_{K,0}^T\right]$. It follows that the $n\times n$ damping, and stiffness matrices of the structure, defined in Eq.(1), can be split as follows:

$$\mathbf{K}(\boldsymbol{\alpha}_K) = \mathbf{K}_\mathrm{S} + \mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_K); \quad \mathbf{C}(\boldsymbol{\alpha}_C) = \mathbf{C}_\mathrm{S} + \mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_C) \tag{3}$$

in which $\mathbf{K}_\mathrm{S}$ and $\mathbf{C}_\mathrm{S}$ are the stiffness and damping matrices of the structure without devices, respectively; $\mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_K) = \mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_{K,0}) + \Delta\mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_K)$ and $\mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_C) = \mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_{C,0}) + \Delta\mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_C)$ are the additional stiffness and damping matrices due to the installation of devices. They are composed by $\mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_{K,0})$ and $\mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_{C,0})$, evaluated in correspondence of the nominal parameter vector $\boldsymbol{\alpha}_0$ of seismic devices, and by $\Delta\mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_C) = \mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_C) - \mathbf{C}_\mathrm{D}(\boldsymbol{\alpha}_{C,0})$ and $\Delta\mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_K) = \mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_K) - \mathbf{K}_\mathrm{D}(\boldsymbol{\alpha}_{K,0})$, their deviations with respect to the additional stiffness and damping matrices evaluated at nominal seismic device parameters.

Due to the presence of seismic devices, the structural system generally could become non-classically damped, it follows that to evaluate the structural response, Eq.(1) has to be written in state-variables:

$$\dot{\mathbf{Z}}(t,\boldsymbol{\alpha}) = \mathbf{D}(\boldsymbol{\alpha})\mathbf{Z}(t,\boldsymbol{\alpha}) + \mathbf{w}\ddot{U}_\mathrm{g}(t); \quad \mathbf{Z}(t_0,\boldsymbol{\alpha}) = \mathbf{0} \tag{4}$$

where $\mathbf{Z}(t,\boldsymbol{\alpha})$ is the 2n-state-variable vector, while the matrix $\mathbf{D}(\boldsymbol{\alpha})$, of order $2n\times 2n$, and the vector $\mathbf{w}$, of order $2n$, are defined, respectively, as:

$$\mathbf{Z}(t,\boldsymbol{\alpha}) = \begin{bmatrix} \mathbf{U}(t,\boldsymbol{\alpha}) \\ \dot{\mathbf{U}}(t,\boldsymbol{\alpha}) \end{bmatrix}; \quad \mathbf{D}(\boldsymbol{\alpha}) = \begin{bmatrix} \mathbf{O}_{n,n} & \mathbf{I}_n \\ -\mathbf{M}^{-1}\mathbf{K}(\boldsymbol{\alpha}_K) & -\mathbf{M}^{-1}\mathbf{C}(\boldsymbol{\alpha}_C) \end{bmatrix}; \quad \mathbf{w} = \begin{bmatrix} \mathbf{O}_{n,1} \\ -\boldsymbol{\tau} \end{bmatrix} \tag{5}$$

with $\mathbf{I}_n$ the n-order identity matrix and $\mathbf{O}_{n,s}$ the zero matrix of order $n\times s$.

2.2 DETERMINISTIC SENSITIVITY FUNCTION

The sensitivity vector of the structural response, $\mathbf{s}_{\mathbf{Z},i}(\boldsymbol{\alpha}_0,t)$, with respect to i-th parameter α_i, i-th element of the r-order parameter vector $\boldsymbol{\alpha}$, is defined as follows:

$$\mathbf{s}_{\mathbf{Z},i}(t,\boldsymbol{\alpha}_0) = \left.\frac{\partial\mathbf{Z}(t,\boldsymbol{\alpha})}{\partial\alpha_i}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} \tag{6}$$

It follows that by performing the differentiation of Eq.(4), with respect to i-th parameter α_i, and setting $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0$, a differential equation governing the evolution of state-variable sensitivity vector is obtained (Cacciola et al, 2005), whose solution can be written as:

$$\mathbf{s}_{\mathbf{Z},i}\left(t,\boldsymbol{\alpha}_0\right)=\int_{t_0}^{t}\boldsymbol{\Theta}\left(t-\tau,\boldsymbol{\alpha}_0\right)\mathbf{D}_i'\left(\boldsymbol{\alpha}_0\right)\boldsymbol{\Psi}(\boldsymbol{\alpha}_0)\mathbf{X}\left(t,\boldsymbol{\alpha}_0\right)\mathrm{d}\tau \tag{7}$$

in which the matrix $\mathbf{D}_i'\left(\boldsymbol{\alpha}_0\right)$ is the differentiation of the matrix $\mathbf{D}\left(\boldsymbol{\alpha}\right)$ with respect to *i*-th parameter, α_i, and setting $\boldsymbol{\alpha}=\boldsymbol{\alpha}_0$; $\boldsymbol{\Theta}\left(\boldsymbol{\alpha},t\right)$ is the transition matrix which can be evaluated once the following eigenproblem is solved:

$$\mathbf{D}^{-1}\left(\boldsymbol{\alpha}\right)\boldsymbol{\Psi}\left(\boldsymbol{\alpha}\right)=\boldsymbol{\Psi}\left(\boldsymbol{\alpha}\right)\boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\alpha}\right);\quad \boldsymbol{\Psi}^T\left(\boldsymbol{\alpha}\right)\mathbf{A}\left(\boldsymbol{\alpha}_C\right)\boldsymbol{\Psi}\left(\boldsymbol{\alpha}\right)=\mathbf{I}_{2m};\quad \mathbf{A}\left(\boldsymbol{\alpha}_C\right)=\begin{bmatrix}\mathbf{C}\left(\boldsymbol{\alpha}_C\right) & \mathbf{M}\\ \mathbf{M} & \mathbf{O}_{n,n}\end{bmatrix}. \tag{8}$$

where the superscript T denotes the transpose operator, $\boldsymbol{\Lambda}(\boldsymbol{\alpha})$ is a diagonal matrix collecting the first $2m$ complex eigenvalues ($m\le n$ is the number of *complex modes* selected for the analysis), and $\boldsymbol{\Psi}(\boldsymbol{\alpha})$ is a complex matrix, of order $(2n\times 2m)$, collecting the corresponding $2m$ complex eigenvectors. Once the eigenproblem (8) is solved, the transition matrix can be evaluated as follows:

$$\boldsymbol{\Theta}\left(t,\boldsymbol{\alpha}\right)=\exp\left[\,t\,\mathbf{D}\left(\boldsymbol{\alpha}\right)\right]=\boldsymbol{\Psi}\left(\boldsymbol{\alpha}\right)\exp\left[t\,\boldsymbol{\Lambda}\left(\boldsymbol{\alpha}\right)\,\right]\boldsymbol{\Psi}^T\left(\boldsymbol{\alpha}\right)\mathbf{A}\left(\boldsymbol{\alpha}_C\right) \tag{9}$$

Alternatively, the state-variable sensitivity vector (7), with respect to the *i*-th parameter, can be evaluated as:

$$\mathbf{s}_{\mathbf{Z},i}\left(t,\boldsymbol{\alpha}_0\right)=\boldsymbol{\Psi}\left(\boldsymbol{\alpha}_0\right)\mathbf{Y}_i\left(t,\boldsymbol{\alpha}_0\right) \tag{10}$$

where $\mathbf{Y}_i\left(t,\boldsymbol{\alpha}_0\right)$ is the sensitivity vector of the response, with respect to the parameter α_i, into the complex modal subspace, given as:

$$\mathbf{Y}_i\left(t,\boldsymbol{\alpha}_0\right)=\int_{t_0}^{t}\exp\left[\left(\tau-\rho\right)\boldsymbol{\Lambda}\left(\boldsymbol{\alpha}_0\right)\right]\mathbf{B}_i\left(\boldsymbol{\alpha}_0\right)\mathbf{X}\left(\tau,\boldsymbol{\alpha}_0\right)\mathrm{d}\tau \tag{11}$$

and:

$$\mathbf{B}_i\left(\boldsymbol{\alpha}_0\right)=\boldsymbol{\Psi}^T\left(\boldsymbol{\alpha}_0\right)\mathbf{A}\left(\boldsymbol{\alpha}_{C,0}\right)\mathbf{D}_i'\left(\boldsymbol{\alpha}_0\right)\boldsymbol{\Psi}\left(\boldsymbol{\alpha}_0\right);\quad \mathbf{v}(\boldsymbol{\alpha}_0)=\boldsymbol{\Psi}^T(\boldsymbol{\alpha}_0)\mathbf{A}(\boldsymbol{\alpha}_{C,0})\mathbf{w} \tag{12}$$

Notice that for deterministic excitation the state-variable sensitivity vectors (7) and (11) with respect to the *i*-th uncertain parameter, can be easily evaluated by step-by-step procedures (Cacciola et al, 2005).

2.3 DEFINITION OF SEISMIC ACCELERATIONS AS FULLY NON-STATIONARY RANDOM PROCESSES

In this paper it is assumed that the ground motion acceleration, $\ddot{U}_{\mathrm{g}}(t)$, is a zero-mean Gaussian fully non-stationary random process. In order to define this process, here the Priestley spectral representation of non-stationary processes is adopted (Priestley, 1965; 1967). Moreover, in the stochastic analysis the one-sided *Power Spectral Density* (*PSD*) function is generally used to characterize the input process. It has been demonstrated that, since the one-sided *PSD* function is not symmetric (Di Paola, 1985; Di Paola and Petrucci, 1990; Muscolino, 1991), the corresponding autocorrelation function is a complex function having real part coincident with the autocorrelation function corresponding to the two-sided *PSD*. This implies that, from a mathematical point of view, the zero-mean Gaussian fully non-stationary random process is a complex process. It can be defined by means of the following *Fourier-Stieltjes integral*:

$$\ddot{U}_{\mathrm{g}}(t)=\sqrt{2}\int_0^{\infty}\exp(\mathrm{i}\omega t)\,a(\omega,t)\,\mathrm{d}N(\omega) \tag{13}$$

where $\mathrm{i}=\sqrt{-1}$ is the imaginary unit; $a(\omega,t)$ is a slowly varying complex deterministic time-frequency modulating function which has to satisfy the condition: $a(\omega,t)\equiv a^*(-\omega,t)$; $N(\omega)$ is a zero-mean process with orthogonal increments satisfying the condition:

$$\mathrm{E}\left\langle \mathrm{d}N(\omega_1)\,\mathrm{d}N^*(\omega_2)\right\rangle=\delta(\omega_1-\omega_2)\,G_0(\omega_1)\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2 \tag{14}$$

where the asterisk * denotes the complex conjugate quantity; the operator $\mathrm{E}\langle\bullet\rangle$ denotes the stochastic average; $\delta(\bullet)$ is the Dirac delta, and $G_0(\omega)$ is the one-sided *PSD* function of the “embedded” stationary counterpart process (Michealov et al, 1999), which is a real function for $\omega\ge 0$, while $G_0(\omega)=0$ for $\omega<0$.

Notice that, since a one-sided *PSD* has been herein assumed, the zero-mean Gaussian non-stationary random process $\ddot{U}_{\mathrm{g}}(t)$ is a complex one (Di Paola, 1985; Di Paola and Petrucci, 1990; Muscolino, 1991; Muscolino and Alderucci, 2015). This process can be completely defined, in the time domain, by the knowledge of its complex autocorrelation function:

$$R_{\ddot{U}_{\mathrm{g}}\ddot{U}_{\mathrm{g}}}(t_1,t_2)\equiv\mathrm{E}\left\langle \ddot{U}_{\mathrm{g}}(t_1)\,\ddot{U}_{\mathrm{g}}(t_2)\right\rangle=\int_0^{\infty}\exp\left[\mathrm{i}\omega(t_1-t_2)\right]a(\omega,t_1)\,a^*(\omega,t_2)G_0(\omega)\,\mathrm{d}\omega \tag{15}$$

The complex process, $\ddot{U}_{\mathrm{g}}(t)$, having complex autocorrelation function (15) has been called *pre-envelope process* by Di Paola (1985). In the Priestley evolutionary process model, the integrand function

$$G_{\ddot{U}_{\mathrm{g}}\ddot{U}_{\mathrm{g}}}(\omega,t)=\left|a(\omega,t)\right|^2 G_0(\omega) \tag{16}$$

is called one-sided *evolutionary power spectral density* (*EPSD*) function of the non-stationary process $\ddot{U}_{\mathrm{g}}(t)$. This processes is called *fully non-stationary* random processes, since both time and frequency content change. If the modulating function is a time dependent function, $a(\omega,t)\equiv a(t)$, the non-stationary process is called *uniformly modulated* (or *quasi-stationary*) random process. In the latter case the *EPSD* function assumes the expression: $G_{\ddot{U}_{\mathrm{g}}\ddot{U}_{\mathrm{g}}}(\omega,t)=a(t)^2G_0(\omega)$.

In this paper the seismic excitations, is modeled as fully non-stationary Gaussian stochastic spectrum compatible processes. It is well known that, on the contrary of stationary random model where the spectrum-compatible *PSD* function is univocally determined, for fully non-stationary random model the *EPSD* function cannot be defined univocally (Cacciola 2010). Here the iterative procedure recently proposed by (Alderucci et al, 2019b) is adopted.

3. Explicit sensitivities of the evolutionary power spectral density response function

3.1 Closed form solution for the sensitivity time-frequency varying response vector function

It is well known that the *time-frequency varying response* (*TFR*) vector function of the response plays a central role in the evaluation of the statistics of the response for both classically and non-classically damped structural systems subjected to fully non-stationary stochastic input. In the presence of the unknown *r*-order parameter vector $\boldsymbol{\alpha}$, the *TFR* vector function of nodal response, $\mathbf{Z}(\omega,t,\boldsymbol{\alpha})$, can be evaluated as follows:

$$\mathbf{Z}(\omega,t,\boldsymbol{\alpha}) = \boldsymbol{\Psi}(\boldsymbol{\alpha})\,\mathbf{X}(\omega,t,\boldsymbol{\alpha}) \tag{17}$$

where $\mathbf{X}(\omega,t,\boldsymbol{\alpha})$ is the *TFR* vector function of the modal complex response, given by:

$$\mathbf{X}(\omega,t,\boldsymbol{\alpha}) = \int_{t_0}^{t} \exp\left[(t-\tau)\boldsymbol{\Lambda}(\boldsymbol{\alpha})\right]\mathbf{v}(\boldsymbol{\alpha})\exp(\mathrm{i}\omega\tau)\,a(\omega,\tau)\,\mathrm{d}\tau \tag{18}$$

This vector function, in the following denoted by the acronym *MTFR* (*modal time-frequency varying response*), can be evaluated in closed form for the commonly adopted modulating functions (Muscolino and Alderucci, 2015; Alderucci and Muscolino, 2018). In particular, here the Spanos and Solomos (1983) model for the fully non-stationary seismic excitation is adopted, whose modulating time-frequency functions can be written as:

$$a(\omega,t) = \varepsilon(\omega)\,(t-t_0)\,\exp\left[-\alpha_a(\omega)(t-t_0)\right]\mathbb{U}(t-t_0); \tag{19}$$

where $\varepsilon(\omega)$ and $\alpha_a(\omega)$ could be complex functions. For quiescent structural systems at time $t_0 = 0$, the vector $\mathbf{X}(\omega,t,\boldsymbol{\alpha})$, defined in Eq.(18), can be evaluated in explicit form as (Muscolino and Alderucci, 2015; Alderucci and Muscolino, 2018):

$$\mathbf{X}(\omega,t,\boldsymbol{\alpha}) = -\varepsilon(\omega)\left\{\exp(-\beta(\omega)\,t)\left[\boldsymbol{\Gamma}^2(\omega,\boldsymbol{\alpha}) + t\,\boldsymbol{\Gamma}(\omega,\boldsymbol{\alpha})\right] - \exp\left[t\,\boldsymbol{\Lambda}(\boldsymbol{\alpha})\right]\boldsymbol{\Gamma}^2(\omega,\boldsymbol{\alpha})\right\}\mathbf{v}(\boldsymbol{\alpha})\,\mathbb{U}(t) \tag{20}$$

where $\beta(\omega) = \alpha_a(\omega) - \mathrm{i}\omega$ and $\boldsymbol{\Gamma}(\omega,\boldsymbol{\alpha})$ is a diagonal matrix defined as:

$$\boldsymbol{\Gamma}(\omega,\boldsymbol{\alpha}) = \left[\boldsymbol{\Lambda}(\boldsymbol{\alpha}) + \beta(\omega)\mathbf{I}_{2m}\right]^{-1}. \tag{21}$$

According to Eqs.(10) and (17), the *sensitivity* of the *TFR* vector function, with respect to *i*-th parameter, can be also evaluated as

$$\mathbf{s}_{\mathbf{Z},i}(\omega,t,\boldsymbol{\alpha}_0) = \boldsymbol{\Psi}(\boldsymbol{\alpha}_0)\,\mathbf{Y}_i(\omega,t,\boldsymbol{\alpha}_0) \tag{22}$$

where $\mathbf{Y}_i(\omega,t,\boldsymbol{\alpha}_0)$ is the *sensitivity* of the *TFR* vector function, with respect to the parameter α_i, projected into the complex modal subspace. It can be evaluated as follows:

$$\mathbf{Y}_i(\omega,t,\boldsymbol{\alpha}_0) = \int_0^t \exp\left[(t-\tau)\boldsymbol{\Lambda}(\boldsymbol{\alpha}_0)\right]\mathbf{B}_i(\boldsymbol{\alpha}_0)\,\mathbf{X}(\omega,\tau,\boldsymbol{\alpha}_0)\,\mathrm{d}\tau. \tag{23}$$

In the following, this vector will be synthetically denoted by the acronym *MSTFR* (*modal sensitivity time-frequency vector*). After very simple algebra it can be shown that this vector function can be evaluated as solution of the following differential equation, with zero start conditions at time $t_0 = 0$:

$$\dot{\mathbf{Y}}_i(\omega,t,\boldsymbol{\alpha}_0) = \boldsymbol{\Lambda}(\boldsymbol{\alpha}_0)\mathbf{Y}_i(\omega,t,\boldsymbol{\alpha}_0) + \mathbf{B}_i(\boldsymbol{\alpha}_0)\mathbf{X}(\omega,\tau,\boldsymbol{\alpha}_0)\mathbb{U}(t-t_0); \quad \mathbf{Y}_i(\omega,0,\boldsymbol{\alpha}_0) = \mathbf{0}. \tag{24}$$

To perform the solution of this set of differential equations the *MTFR* vector function vector, defined in Eq.(20), is rewritten as (Alderucci et al, 2019a):

$$\mathbf{X}(\omega,t,\boldsymbol{\alpha}_0) = \mathbf{X}_1(\omega,t,\boldsymbol{\alpha}_0) + \mathbf{X}_2(\omega,t,\boldsymbol{\alpha}_0) \tag{25}$$

where:

$$\begin{aligned} &\mathbf{X}_1(\omega,t,\boldsymbol{\alpha}_0) = -\varepsilon(\omega)\exp(-\beta(\omega)\,t)\,\left[\boldsymbol{\Gamma}_0^2(\omega) + t\,\boldsymbol{\Gamma}_0(\omega)\right]\mathbf{v}_0\,\mathcal{U}(t);\\ &\mathbf{X}_2(\omega,t,\boldsymbol{\alpha}_0) = \varepsilon(\omega)\exp(t\,\boldsymbol{\Lambda}_0)\,\boldsymbol{\Gamma}_0^2(\omega)\mathbf{v}_0\,\mathcal{U}(t). \end{aligned} \tag{26}$$

Since the *MTFR* vector function has been split as the sum of two contributions, the *MSTFR* vector, solution of Eq.(24), can be split as the sum of two vectors too, solutions of the following two sets of differential equations, with zero start initial conditions at time $t_0 = 0$:

$$\begin{aligned} \mathbf{Y}_i(\omega,t,\boldsymbol{\alpha}_0) = \mathbf{Y}_{i,1}(\omega,t,\boldsymbol{\alpha}_0) + \mathbf{Y}_{i,2}(\omega,t,\boldsymbol{\alpha}_0) = \big\{&\mathbf{Y}_{i,1,\mathrm{p}}(\omega,t,\boldsymbol{\alpha}_0) + \mathbf{Y}_{i,2,\mathrm{p}}(\omega,t,\boldsymbol{\alpha}_0)\\ &-\exp(t\,\boldsymbol{\Lambda}_0)\left[\mathbf{Y}_{i,1,\mathrm{p}}(\omega,0,\boldsymbol{\alpha}_0) + \mathbf{Y}_{i,2,\mathrm{p}}(\omega,0,\boldsymbol{\alpha}_0)\right]\big\}; \quad t > 0 \end{aligned} \tag{27}$$

where the particular solution vectors of Eq.(24), can be evaluated, after some algebra, as follows:

$$\begin{aligned} &\mathbf{Y}_{i,1,\mathrm{p}}(\omega,t,\boldsymbol{\alpha}_0) = \varepsilon(\omega)\exp(-\beta(\omega)\,t)\,\boldsymbol{\Gamma}(\omega,\boldsymbol{\alpha}_0)\left[\boldsymbol{\Gamma}(\omega,\boldsymbol{\alpha}_0)\mathbf{B}_i(\boldsymbol{\alpha}_0) + \mathbf{B}_i(\boldsymbol{\alpha}_0)\boldsymbol{\Gamma}(\omega,\boldsymbol{\alpha}_0) + t\,\mathbf{B}_i(\boldsymbol{\alpha}_0)\right]\boldsymbol{\Gamma}(\omega,\boldsymbol{\alpha}_0)\mathbf{v}(\boldsymbol{\alpha}_0);\\ &\mathbf{Y}_{i,2,\mathrm{p}}(\omega,t,\boldsymbol{\alpha}_0) = \varepsilon(\omega)\mathbf{P}_i(t,\boldsymbol{\alpha}_0)\exp\left[t\,\boldsymbol{\Lambda}(\boldsymbol{\alpha}_0)\right]\boldsymbol{\Gamma}^2(\omega,\boldsymbol{\alpha}_0)\mathbf{v}(\boldsymbol{\alpha}_0); \end{aligned} \tag{28}$$

where $\mathbf{P}_i(\boldsymbol{\alpha}_0,t)$ is a matrix of order $(2m\times 2m)$ whose elements, $P_{i,jk}(\boldsymbol{\alpha}_0,t)$, are defined as follows:

$$P_{i,jj}(t,\boldsymbol{\alpha}_0) = t\,B_{i,jj}(\boldsymbol{\alpha}_0); \quad P_{i,jk}(t,\boldsymbol{\alpha}_0) = \frac{B_{i,jk}(\boldsymbol{\alpha}_0)}{\lambda_k - \lambda_j},\ j \neq k \tag{29}$$

with $B_{i,jk}(\boldsymbol{\alpha}_0)$ elements of the matrix $\mathbf{B}_i(\boldsymbol{\alpha}_0)$.

3.3 CLOSED FORM SOLUTIONS FOR THE SENSITIVITIES OF THE EPSD RESPONSE MATRIX FUNCTION

Since it has been assumed that the ground motion acceleration, $\ddot{U}_{\mathrm{g}}(t)$, is a zero-mean Gaussian fully non-stationary random process, with one-sided *EPSD* function, $G_{\ddot{U}_{\mathrm{g}}\ddot{U}_{\mathrm{g}}}(\omega,t)$, the stochastic response is a zero-mean fully non-stationary stochastic vector process too, whose one-sided *EPSD* matrix function, $\mathbf{G_{ZZ}}(\omega,t,\boldsymbol{\alpha})$, can be evaluated as follows (Alderucci et al, 2019a):

$$\mathbf{G_{ZZ}}(\omega,t,\boldsymbol{\alpha}) = G_0(\omega)\,\mathbf{Z}^*(\omega,t,\boldsymbol{\alpha})\mathbf{Z}^T(\omega,t,\boldsymbol{\alpha}) = G_0(\omega)\,\boldsymbol{\Psi}^*(\boldsymbol{\alpha})\mathbf{X}^*(\omega,t,\boldsymbol{\alpha})\mathbf{X}^T(\omega,t,\boldsymbol{\alpha})\boldsymbol{\Psi}^T(\boldsymbol{\alpha}) \tag{30}$$

where $G_0(\omega)$ is the one-sided *PSD* function of the "embedded" stationary counterpart of the input process, while $\mathbf{Z}(\omega,t,\boldsymbol{\alpha})$ and $\mathbf{X}(\omega,t,\boldsymbol{\alpha})$ are the *TFR* vector responses, in state variables, into nodal and modal complex spaces, respectively.

Once the one-sided *EPSD* matrix function, $\mathbf{G}_{\mathbf{ZZ}}(\omega,t)$, is defined, it is possible to evaluate in compact form the statistics of the response as follows:

$$\boldsymbol{\Sigma}_{\mathbf{ZZ}}(t,\boldsymbol{\alpha}) = \boldsymbol{\Psi}^*(\boldsymbol{\alpha})\left[\int_0^\infty G_0(\omega)\,\mathbf{X}^*(\omega,t,\boldsymbol{\alpha})\,\mathbf{X}^T(\omega,t,\boldsymbol{\alpha})\,\mathrm{d}\omega\right]\boldsymbol{\Psi}^T(\boldsymbol{\alpha}) \tag{31}$$

This matrix is the so-called *pre-envelope covariance* (*PEC*) matrix function, in nodal space, it is a $2n\times 2n$ Hermitian matrix, whose real part coincides with the classical covariance matrix (Di Paola, 1985). By differentiating the *PEC* matrix function with respect to the *i*-th parameter, it is possible to evaluate its *sensitivity* function in the neighbourhood of nominal parameters, $\boldsymbol{\alpha}=\boldsymbol{\alpha}_0$ as follows:

$$\boldsymbol{\Sigma}_{\mathbf{s}_{\mathbf{Z},i}}(t,\boldsymbol{\alpha}_0) = \left.\frac{\partial \boldsymbol{\Sigma}_{\mathbf{ZZ}}(t,\boldsymbol{\alpha})}{\partial \alpha_i}\right|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0} = \mathrm{E}\left\langle \mathbf{Z}(t,\boldsymbol{\alpha}_0)\mathbf{s}_{\mathbf{Z},i}^{*T}(t,\boldsymbol{\alpha}_0)\right\rangle + \mathrm{E}\left\langle \mathbf{Z}(t,\boldsymbol{\alpha}_0)\mathbf{s}_{\mathbf{Z},i}^{*T}(t,\boldsymbol{\alpha}_0)\right\rangle^{*T} \tag{32}$$

whose elements are the *sensitivity* of first three spectral moments with respect to the parameter α_i. In the previous equation the sensitivity vector $\mathbf{s}_{\mathbf{Z},i}(t,\boldsymbol{\alpha}_0)$ which has been defined in Eq.(7) appears. It follows that the following relationship holds:

$$\mathrm{E}\left\langle \mathbf{Z}(t,\boldsymbol{\alpha}_0)\mathbf{s}_{\mathbf{z},i}^{*T}(t,\boldsymbol{\alpha}_0)\right\rangle = \boldsymbol{\Psi}^*(\boldsymbol{\alpha}_0)\left\{\int_0^\infty \mathbf{X}^*(\omega,t,\boldsymbol{\alpha}_0)\mathbf{Y}_i^T(\omega,t,\boldsymbol{\alpha}_0)\,G_0(\omega)\mathrm{d}\omega\right\}\boldsymbol{\Psi}^T(\boldsymbol{\alpha}_0) \tag{33}$$

where the vector $\mathbf{X}(\omega,t,\boldsymbol{\alpha}_0)$ and $\mathbf{Y}_i(\omega,t,\boldsymbol{\alpha}_0)$ are given in explicit form in Eqs. (20) and (27), respectively. Substituting Eq.(33) into Eq.(32) the so-called *sensitivity* of the *PEC* matrix function can be rewritten as:

$$\boldsymbol{\Sigma}_{\mathbf{s}_{\mathbf{Z},i}}(t,\boldsymbol{\alpha}_0) \equiv \int_0^\infty \mathbf{G}_{\mathbf{s}_{\mathbf{Z},i}}(\omega,t,\boldsymbol{\alpha}_0)\,\mathrm{d}\omega \tag{34}$$

where the matrix

$$\mathbf{G}_{\mathbf{s}_{\mathbf{Z},i}}(\omega,t,\boldsymbol{\alpha}_0) = G_0(\omega)\,\boldsymbol{\Psi}^*(\boldsymbol{\alpha}_0)\left[\mathbf{X}^*(\omega,t,\boldsymbol{\alpha}_0)\,\mathbf{Y}_i^T(\omega,t,\boldsymbol{\alpha}_0) + \mathbf{Y}_i^*(\omega,t,\boldsymbol{\alpha}_0)\,\mathbf{X}^T(\omega,t,\boldsymbol{\alpha}_0)\right]\boldsymbol{\Psi}^T(\boldsymbol{\alpha}_0). \tag{35}$$

can be interpreted as the *sensitivity* of the *EPSD* matrix function. This matrix function is evaluated by means of closed form solutions too.

4. Numerical Application

In order to show the effectiveness of the proposed method, four Single-Degree-of-Freedom (SDOF) systems are analysed. The selected time-frequency modulating function for the fully non stationary process is the one proposed by Spanos and Solomos (1983), defined in Eq.(19), whose parameters herein selected are:

$$\alpha_a(\omega)=\frac{1}{2}\left(0.15+\frac{\omega^2}{225\pi^2}\right);\quad \varepsilon(\omega)=\frac{\sqrt{2}}{15\pi\, a_{\max}}\omega;\quad t_0=0 \tag{36}$$

where, in order to normalize to one the modulating function, the parameter $a_{\max}$ is herein set equal to 1.34.

For the SDOF system the *PEC* matrix function, in nodal space, it is a 2×2 Hermitian matrix, whose real part coincides with the classical covariance matrix. It can be evaluated as:

$$\mathbf{\Sigma_{zz}}(t,\boldsymbol{\alpha})=\begin{bmatrix}\lambda_{0,u}(t,\boldsymbol{\alpha}) & \mathrm{i}\,\lambda_{1,u}(t,\boldsymbol{\alpha})\\ -\mathrm{i}\,\lambda^{*T}_{1,u}(t,\boldsymbol{\alpha}) & \lambda_{2,u}(t,\boldsymbol{\alpha})\end{bmatrix} \tag{37}$$

where the function $\lambda_{i,u}(t,\boldsymbol{\alpha})$ is the so-called *non-geometric spectral moments* (*NGSM*) of *i*-th order of stochastic response (Michealov et al, 1999). The *EPSD* function, having the spectrum-compatible stationary counterpart *PSD* evaluated according to Alderucci et al. (2019b), is depicted in Figure 1.

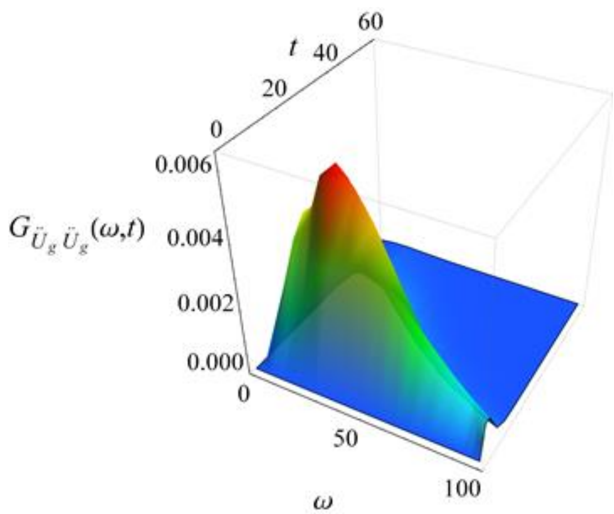

Figure 1. EPSD function of the input process according to Alderucci et al. (2019b) model.

The four SDOF systems have damping ratio equal to $\zeta_0=0.02$ and natural periods $T_0=0.1$ s, $T_0=0.2$ s, $T_0=0.6$ s, and $T_0=1$ s, respectively. A damper device, with damping coefficient c_d and stiffness k_d is connected to each SDOF system. The analysis is conducted varying the abovementioned damper parameters within $1\div10^8$ Ns/m and $1\div10^8$ N/m, respectively, in order to obtain the optimal design values.

Figure 2 depicts, for each SDOF system, the maximum values of the first NGSM, $\max\left\{\lambda_{0,u}(t)\right\}$, that is coincident with the maximum value of the variance of the response, versus the damper stiffness, for different values of the damping coefficients. From the analysis of these figures, it can be evidenced that changes in stiffness values don't modify significantly the response. Consequently, according to the usual characteristics of the commercial devices, the design stiffness value can be assumed equal to 30 N/m for all the analysed oscillators.

In Figure 3, for the four oscillators, the maximum values of the variance of the response, versus the damping coefficient, for the previously chosen optimal value of the stiffness $k_d = 30\,\mathrm{N/m}$, is reported.

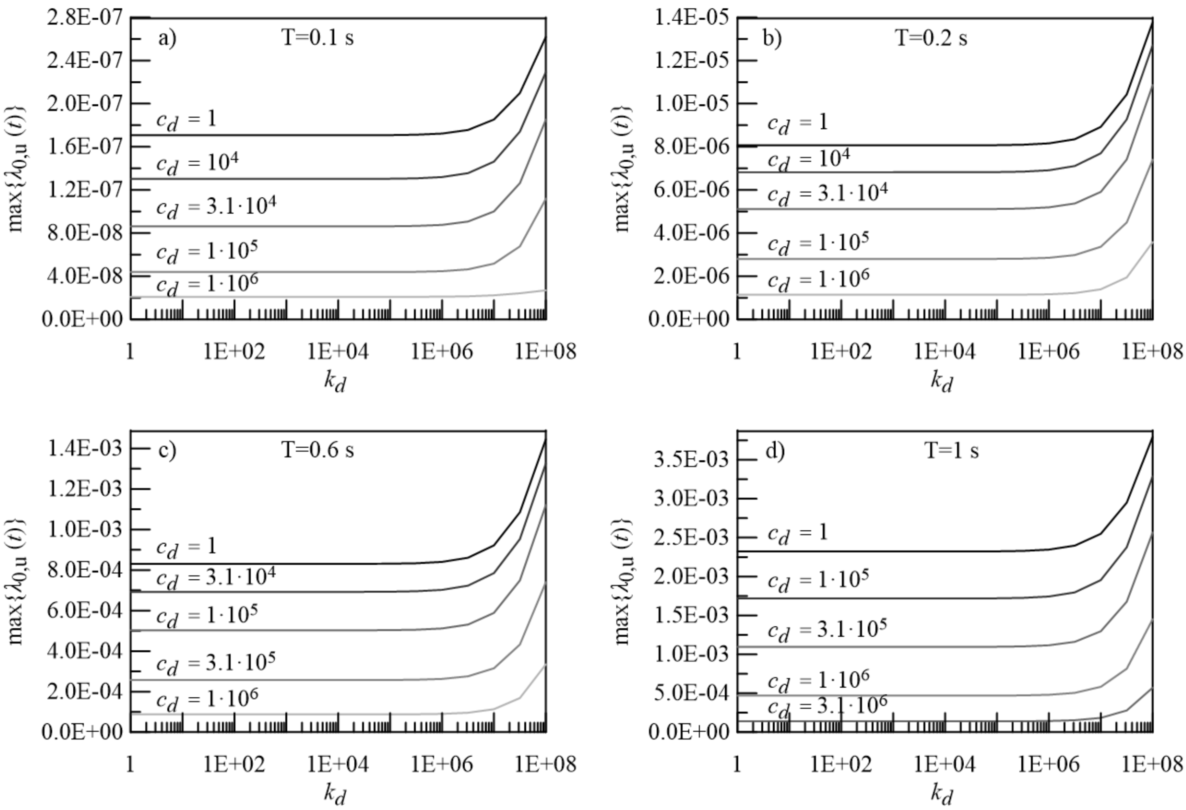

Figure 2. Maximum values of the variance of the response, $\max\left\{\lambda_{0,u}(t)\right\}$, for the four oscillators, versus the stiffness of the device, varying the damping coefficient: a) $T_0 = 0.1$ s, b) $T_0 = 0.2$ s, c) $T_0 = 0.6$ s, d) $T_0 = 1.0$ s.

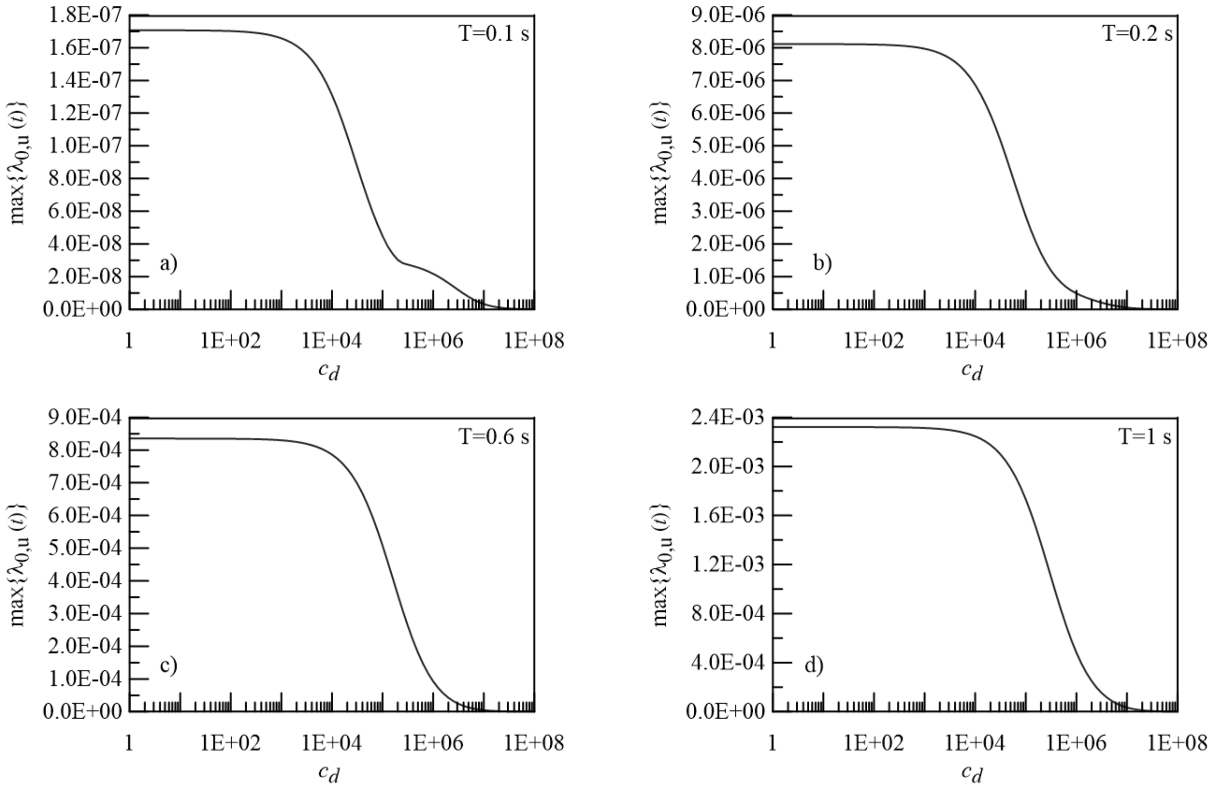

Figure 3. Maximum values of the variance of the response, $\max\left\{\lambda_{0,u}(t)\right\}$, for the four oscillators, versus the damping coefficient of the device: a) $T_0 = 0.1$ s, b) $T_0 = 0.2$ s, c) $T_0 = 0.6$ s, d) $T_0 = 1.0$ s.

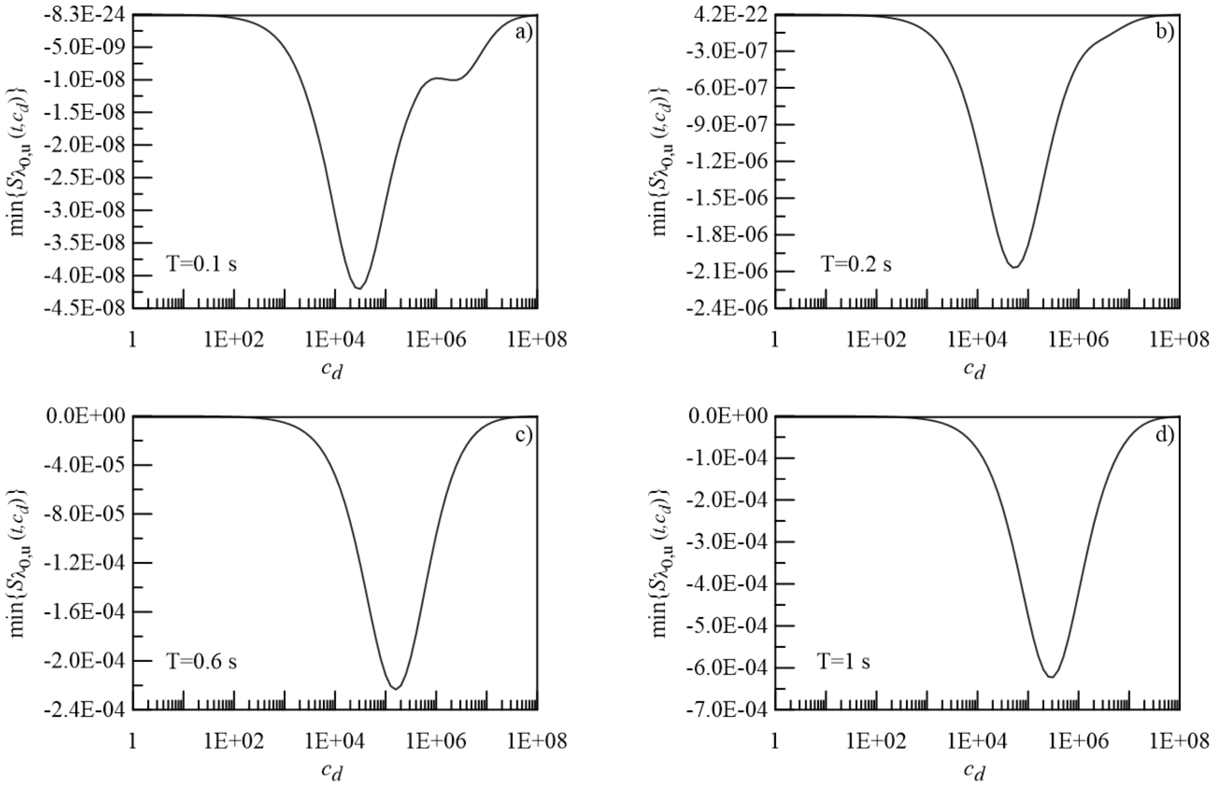

Figure 4. Minimum values of the sensitivity function of the variance of the response versus the damping coefficient , $\min\left\{S_{\lambda_0}(t,c_d)\right\}$, varying the damping coefficient of the device: a) $T_0 = 0.1$ s, b) $T_0 = 0.2$ s, c) $T_0 = 0.6$ s, d) $T_0 = 1.0$ s.

Figure 4 depicts the minimum values of the sensitivity of the variance of the response $\min\left\{S_{\lambda_0}(t,c_d)\right\} = \min\left\{\left|\partial\lambda_{0,u}(t)/\partial c_d\right|_{k_d=k_{d,0}}\right\}$, for the four analysed SDOF versus the damping coefficient of the damper device.

It is well known that for small variation of a parameter with respect the nominal one, it is possible to predict with good accuracy the variation of the response spectral moment by the knowledge of its sensitivity. Then, for negative sensitivities, the optimal value can be obtained when the sensitivity reaches its minimum value. It follows that, for the four SDOF systems, the optimal damping coefficients, reported in Table 1, are those corresponding to the points at which the minimum values of the sensitivity functions, depicted in Figure 4, assume the smallest values. Therefore, they can be selected as the optimal damping coefficients for the design dissipation devices.

Table I. Optimal damping values for each SDOF system	
T [s]	c_d [Ns/m]
0.1	31 623
0.2	50 119
0.6	158 489
1.0	310 000

5. Summary and Conclusions

The present work aimed to define a new method to evaluate sensitivities of stochastic response characteristics of structural systems subjected to seismic excitations; the ground motion acceleration was herein modeled as fully non-stationary Gaussian stochastic processes.

Closed form solutions for the first-order derivatives of the *TFR* as well as of the one-sided evolutionary *PSD* (*EPSD*) of the structural response, with respect to damping parameters of devices, are evaluated. Numerical applications on different SDOF oscillators showed the accuracy and the computational efficiency of the proposed method.

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