# An Evolutive Probability Transformation Method for the Dynamic Stochastic Analysis of Structures

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**Abstract.** The problem of the stochastic response determination of dynamic systems, once that the random properties of the actions are assigned, is studied. In the present work, an extension of the Probability Transformation Method (PTM) to the dynamic systems has been proposed. In particular, an evolutive PTM (EPTM) is introduced. Based on the properties of the mean square random calculus with the principle of conservation of probability, the EPTM applies the PTM contextually to the mean square Riemann integral of the system solution process. In this way, the EPTM gives the expression of the single-time evolutive characteristic function (CF) of the system's output. The application to a shear-type plane system subject to assigned stochastic process excitations and random initial conditions has revealed the efficiency of the stochastic procedure.

**Keywords:** Evolutive Probability Transformation Method, stochastic dynamic analysis, random process

## 1. Introduction

Over the last two decades, it has been recognized the uncertain character in the natural phenomena. Most physical behaviors exhibit appreciable randomness which can not be adequately represented by deterministic models. Stochastic differential equations are often used to model the stochastic dynamics of uncertain systems. Hence there is the need to carry out a stochastic analysis that describes the response in probabilistic terms. Efficient probabilistic characterization of the dynamical response of a system excited by random actions often requires a high computational effort. In the last fifty years, many significant results have been obtained in this field. Most of the approaches in the literature consider the solving input-output relations in terms of evolutive moments, or cumulants, or quasi-moments, or, for having a multiple times definition, in terms of correlation functions of various order (Lin, 1967; Roberts and Spanos, 1991; Wu and Lin, 1984; Lutes and Sarkani, 2004; Di Paola *et al.*, 1992; Di Paola and Falsone, 1994; Falsone, 1994; Di Paola and Falsone, 1997a,b; Falsone, 2005; Morikawa and Kameda, 1997; Makarios, 2012; Gioffrè and Gusella, 2002; Mazelsky, 1954; Bucher and Schueller, 1991). Unfortunately, all these quantities suffer the drawback of having high dimensions, above all for large systems.

In some problems, such as the structural reliability evaluation or the stochastic limit analysis, the accurate knowledge of the response PDF is essential, above all at the PDF tails, (Di Paola and Falsone, 1994; Falsone, 1994). However, only a few works perform dynamic stochastic analyses in terms of probability density function (PDF) (Conte and Peng, 1996; Adhikari, 2007; Hussein and Selim, 2015; Li, 2016; Calatayud *et al.*, 2018b,a; Liu and Liu, 2018; Kalogeris and Papadopoulos,

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2018; Meimaris *et al.*, 2019; Mamis *et al.*, 2019). In particular, the present authors have extended to the dynamic analysis an approach, previously introduced for static analyses and called probability transformation method (PTM) (Falsone and Settineri, 2013a,b). This last one is essentially based on the rules of random variable transformations and on the principle of probability conservation. The above-cited extension to the dynamic analyses was made through a time discretization in the integral expression of the structural response (Falsone and Laudani, 2018, 2020). In this way, the input-output relationships are considered as algebraic equations and the PTM can be applied as in the static analyses.

In the present work, a study on the evaluation of the response of dynamic systems governed by first-order differential equations will be presented. Problems described by this type of equations arise in a wide variety of applied engineering areas. Within the framework of the mean square random calculus, a stochastic procedure that combines the properties of the PTM with an approach able to find numerically the relationship between a process and its time integral by working in terms of characteristic functions (CF) will be developed. Therefore, an evolutive PTM (EPTM) will be is introduced. In this way, while in the previous approach the PTM is applied after the numerical discretization (Falsone and Laudani, 2018, 2020), the EPTM applies the PTM contextually to the integral procedure. This makes the EPTM a very efficient approach for the dynamic stochastic analyses of systems described by first-order differential systems, as has been verified in a shear-type plane system that will be presented. In particular, this work performs the response PDF considering the following two stochastic actions of the system, (i) time-dependent actions represented as random processes, (ii) system initial conditions represented as random variables. For both these two possibilities, which could happen contemporaneously, the expression of the single-time varying CF output of a stochastic dynamic system has been obtained.

## 2. Preliminary concepts

In this section, some basic concepts, which will be useful in the following sections, are shown. In particular, the PTM is recalled.

The fundamental aspects of the PTM can be found in the theory of the space transformation of random vectors as well as in the *principle of probability conservation.* It is possible to state the principle of probability conservation as following: *the probability carried by a random event is conserved without introducing other stochastic factors. In other words, if the random factors involved in a stochastic system are preserved, i.e no new random factors arise nor existing factors vanish in a physical process, then the probability will be preserved in the evolution process of the system* (Li and Chen, 2009; Soong, 1973). In particular, the PTM allows working directly in terms of input and output probability density functions (PDFs) of two random vectors related to each other by the deterministic law corresponding to the assigned space transformation.

Let consider a $n$-dimensional random vector, $\mathbf{x}$, with joint probability density function (JPDF), $p_{\mathbf{x}}(\mathbf{x})$, and let $\mathbf{h}(\cdot)$ be a $n$-dimensional invertible application, with inverse $\mathbf{h}(\cdot)^{-1} = \mathbf{g}(\cdot)$. If $\mathbf{z} = \mathbf{h}(\mathbf{x})$ is a random vector whose JPDF, under the respect of the principle of probability conservation, can be expressed as follows:

$$p_{\mathbf{z}}(\mathbf{z}) = \frac{1}{|\det[\mathbf{J_h}(\mathbf{g}(\mathbf{z}))]|} p_{\mathbf{x}}(\mathbf{g}(\mathbf{z})) = |\det[\mathbf{J_g}(\mathbf{z})]|\, p_{\mathbf{x}}(\mathbf{g}(\mathbf{z})) \tag{1}$$

where $\mathbf{J_h}(\cdot)$ and $\mathbf{J_g}(\cdot) = \mathbf{J_h}^{-1}(\cdot)$ are the Jacobian matrices corresponding to the relation $\mathbf{z} = \mathbf{h}(\mathbf{x})$ and the inverse $\mathbf{x} = \mathbf{g}(\mathbf{z})$. The relationship in Eq. (1) holds when $\mathbf{x}$ and $\mathbf{z}$ have the same number of components. Nevertheless, this is not a restriction, in Falsone and Laudani (2019a) some procedures are given to apply it even when these numbers of components are different.
In Falsone and Settineri (2013a,b) it was introduced an efficient way to evaluate the marginal PDF of $\mathbf{z}$, $p_{z_j}(z_j)$. By using the properties of the multidimensional Dirac delta function, $\delta(\cdot)$ Eq. (1) is rewritten in the following form:

$$p_{\mathbf{z}}(\mathbf{z}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\mathbf{x}}(\mathbf{y}) \delta(\mathbf{z} - \mathbf{h}(\mathbf{y})) dy_1 \cdots dy_n. \tag{2}$$

The use of the latter expression is particularly useful if the marginal PDF evaluation of $\mathbf{z}$ is required. For example, if only the element $z_j = h_j(\mathbf{x})$ has to be studied, then Eq. (1) gives:

$$p_{z_j}(z_j) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\mathbf{x}}(\mathbf{y}) \delta(z_j - h_j(\mathbf{y})) dy_1 \cdots dy_n \tag{3}$$

while, if the JPDF of the elements $z_j = h_j(\mathbf{x})$ and $z_k = h_k(\mathbf{x})$ has to be evaluated, then the following expression is suitable:

$$p_{z_j, z_k}(z_j, z_k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\mathbf{x}}(\mathbf{y}) \delta(z_j - h_j(\mathbf{y})) \delta(z_k - h_k(\mathbf{y})) dy_1 \cdots dy_n. \tag{4}$$

It is worth noting that Eqs. (2) to (4) do not require the evaluation of the inverse relation $\mathbf{g}(\cdot) = \mathbf{h}(\cdot)^{-1}$, that often can be a great problem. Moreover, the order of the vector $\mathbf{z}$ does not influence them. On the contrary, they have the drawback of requiring $n$ integrations respect to the component of $\mathbf{x}$. This last problem can be easily solved in terms of characteristic function (CF) when $h_j(\mathbf{x})$ is given as a linear combination of the elements of $\mathbf{x}$. This means the relation $h_j(\mathbf{x}) = \mathbf{h}_j^T \mathbf{x}$, where $\mathbf{h}_j$ is the $n$-vector made by the combination constants. Indeed, the CF of $z_j$ is given by:

$$\begin{aligned} M_{z_j}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} p_{z_j}(z_j) \exp(-i\omega z_j) dz_j \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\mathbf{x}}(\mathbf{y}) \delta(z_j - \mathbf{h}_j^T \mathbf{y}) dy_1 \cdots dy_n \right] \exp(-i\omega z_j) dz_j. \end{aligned} \tag{5}$$

that, taking into account the properties of the Dirac delta function, gives:

$$M_{z_j}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{\mathbf{x}}(\mathbf{y}) \exp(-i\omega \mathbf{h}_j^T \mathbf{y}) dy_1 \cdots dy_n = (2\pi)^{n-1} M_{\mathbf{x}}(\boldsymbol{\theta})|_{\boldsymbol{\theta} = \omega \mathbf{h}_j}. \tag{6}$$

This expression evidences the fundamental result that the response CF, $M_{z_j}(w_j)$, is always obtainable in closed form, once that the multidimensional CF of the input is known, without the necessity of any integration. If the joint CF (JCF) of the output random variables $z_j$ and $z_k$ has to be evaluated, then the following expression, easily obtained by generalizing the previous one can be used:

$$M_{z_j z_k}(\omega_j, \omega_k) = (2\pi)^{n-2} M_{\mathbf{X}}(\boldsymbol{\theta})|_{\boldsymbol{\theta}=\omega_j \mathbf{h}_j + \omega_k \mathbf{h}_k} \tag{7}$$

The generalization to greater order JCFs is immediate. Once that the characteristic functions are evaluated the corresponding PDF can be obtained by Fourier anti-transform operations.

Therefore, from this section, it is worth highlighting the following essential concept: *in a generic stochastic system if no other stochastic factors are involved then the probability carried by a random event is conserved.*

## 3. Stochastic first-order differential equation systems

A large class of physical behaviors and natural phenomena can be described by ordinary differential equations. Taking into account the effects of the various inherent uncertainties in nature when the dynamic analysis of deterministic systems is performed, it may arise the necessity to perform a stochastic analysis. It is clear that although a generic system of equations involves a set of higher-order differential equations, it is always possible, by introducing more variables, to convert higher-order differential equations to a set of coupled first-order equations. In view of these considerations, in this section, stochastic first-order differential systems will be treated. In general, the uncertain behavior of the system is due to (i) time-dependent actions represented as random processes, (ii) system initial conditions represented as random variables. Both these possibilities, which could happen contemporaneously, will be investigated in the following subsections, respectively.

### 3.1. FIRST-ORDER DIFFERENTIAL SYSTEMS EXCITED BY A GIVEN RANDOM PROCESS

Let consider a first-order differential system governed by the following equation:

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{F}(t); \quad \mathbf{X}(t_0) = \mathbf{X}_0 \tag{8}$$

where $\mathbf{X}(t)$ is the $n$-vector of the response state variables, $\mathbf{A}$ is the $n \times n$ deterministic matrix which takes into account the physical-geometrical characteristics of the system, $\mathbf{B}$ is the $n \times m$ matrix defining the distribution of the external loads on the system, $\mathbf{F}(t)$ is the $m$-random vector whose probabilistic characterization is known and, at last, the vector $\mathbf{X}_0$ defines the initial conditions of the system that, here, are assumed to be deterministic. Assuming that the m.s. derivative $\dot{\mathbf{X}}(t)$ of $\mathbf{X}(t)$ exists, the solution of Eq. (8) is well known:

$$\mathbf{X}(t) = \boldsymbol{\Theta}(t - t_0)\mathbf{X}_0 + \int_{t_0}^{t} \boldsymbol{\Theta}(t - \tau)\mathbf{B}\mathbf{F}(\tau)d\tau \tag{9}$$

$\boldsymbol{\Theta}(t) = \exp(-\mathbf{A}t)$ being the fundamental matrix of the system. The first of the two quantities composing $\mathbf{X}(t)$ is deterministic, under the assumptions made here, while the second one defines a vector random process. Hence, it is useful to write:

$$\mathbf{X}(t) = \mathbf{Y}_0(t) + \mathbf{Y}(t); \quad \text{with} \quad \mathbf{Y}_0(t) = \boldsymbol{\Theta}(t - t_0)\mathbf{X}_0, \quad \mathbf{Y}(t) = \int_{t_0}^{t} \boldsymbol{\Theta}(t - \tau)\mathbf{B}\mathbf{F}(\tau)d\tau. \tag{10}$$

By assuming zero initial conditions, the attention is paid only on the vector random process $\mathbf{Y}(t)$ and on its relationship with the load random process $\mathbf{F}(t)$.
If $\mathbf{F}(t)$ is a Gaussian vector process, then it is simple to show that $\mathbf{Y}(t)$ is Gaussian, too. Hence, it is characterized by the knowledge of its mean vector and its cross-covariance matrix. When $\mathbf{F}(t)$ is not-Gaussian, the probabilistic characterization of the response becomes more complicated. At this scope, the following vector process is introduced:

$$\mathbf{Z}(\tau)=\mathbf{\Theta}(t-\tau)\mathbf{B}\mathbf{F}(\tau) \tag{11}$$

and Eq. (10) is rewritten as:

$$\mathbf{Y}(t)=\int_{t_0}^{t}\mathbf{\Theta}(t-\tau)\mathbf{B}\mathbf{F}(\tau)d\tau=\int_{t_0}^{t}\mathbf{Z}(\tau)d\tau. \tag{12}$$

Its JPDF $p_{\mathbf{Z}}(\mathbf{z},\tau)$ can be easily obtained through the use of the PTM rules to the linear transformation $\mathbf{F}(\tau)\to\mathbf{Z}(\tau)$. The last step to do is the characterization of the random vector process $\mathbf{Y}(t)$, once that the characterization of the vector process $\mathbf{Z}(t)$ has been found. Due to the time integral relationship between these two processes, an efficient numerical solution can be obtained by using the following theorem on the JCFs of $\mathbf{Y}(t)$ and $\mathbf{Z}(t)$:

THEOREM 1. *((Soong, 1973) If the m.s. integral* $\mathbf{X}(t)$, $t\in T$, *exists, then:*

$$M_{\mathbf{Y}}(\omega,t)=\lim_{m\to\infty\Delta m\to 0} M_{\mathbf{Z}}(\omega\left(\tau_1-\tau_0\right),\tau_1';\cdots;\omega\left(\tau_m-\tau_{m-1}\right),\tau_m') \quad \tau_j'\in\left(\tau_{j-1},\tau_j\right) \tag{13}$$

$\Delta_m=\max\left(\tau_{j-1}-\tau_j\right)$ .

The result of this theorem is mainly based on the mean square calculus as well as on the principle of conservation of probability. In fact, employing some of the theory of random differential equations and assuming that $\mathbf{Y}(t)$ is a mean square Riemann integral over a generic interval $[a,b]$, it can be written:

$$\int_a^b\mathbf{Y}(t)dt\equiv\lim_{\Delta m\to 0}\sum_{j=1}^{m}\mathbf{Y}(t_j')\Delta t_j \tag{14}$$

where a collection of all finite partitions of the interval $[a,b]$ was considered as in Figure 1. Then, in accordance with the principle of conservation of probability, the formal relationship between the characteristic functions of $\mathbf{Y}(t)$ and that of $\mathbf{Z}(t)$ (Eq. (13)) can be obtained. Therefore, by truncating the value of $m$ to a sufficiently high value, the expression of this theorem becomes an efficient way to evaluate numerically the response CF. Then, the response PDF is obtained by the Fourier anti-transform of the response CF. Moreover, if the characterization of the response random process is required at more instant, Eq. (13) must be generalized. This is always possible, even if the corresponding numerical evaluation becomes more and more heavy increasing the number of time instants.

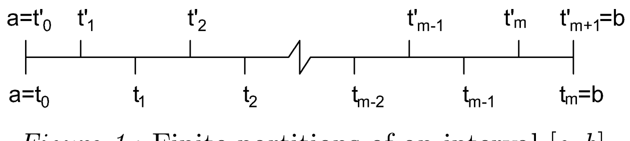


*Figure 1.*: Finite partitions of an interval $[a,b]$.

Overall, the application of the PTM with the fundamental results of the above Theorem 1 gives a stochastic procedure, here called EPTM, able to characterize the response of dynamical systems in terms of evolutive PDF.

### 3.2. FIRST-ORDER DIFFERENTIAL SYSTEMS WITH RANDOM LOAD AND RANDOM INITIAL CONDITIONS

Here, besides the loads acting on the system, even the initial conditions are assumed to be random variables. In particular, in this section, the EPTM approach in the assumption of independence between these two kinds of actions will be developed. In taking the expression of the solution $\mathbf{X}(t)$ given in Eq. (9), there is no need to assume the independence of $\mathbf{F}(\tau)$ and $\mathbf{X}_0$, but the initial condition is generally independent of the forcing term in practice. The expression of the response vector can be rewritten as follows:

$$\mathbf{X}(t) = \bar{\mathbf{X}}_0(t) + \bar{\mathbf{X}}_F(t) \tag{15}$$

$\mathbf{X}_0$ and $\bar{\mathbf{X}}_F(t)$ being two independent random processes given by:

$$\begin{aligned} \bar{\mathbf{X}}_0(t) &= \mathbf{\Theta}(t)\mathbf{X}_0; \\ \bar{\mathbf{X}}_F(t) &= \int_0^t \mathbf{\Theta}(t-\tau)\mathbf{B}\mathbf{F}(\tau)d\tau \end{aligned} \tag{16}$$

Each of these processes can be easily characterized through the evaluation of their CFs. Indeed, $\bar{\mathbf{X}}_0(t)$ is a linear combination of the elements of $\bar{\mathbf{X}}_F(t)$ and, thus, its CF can be evaluated in the form given in Eq. (6). The process $\bar{\mathbf{X}}_F(t)$ is the time integral of the process $\mathbf{Z}(\tau) = \mathbf{\Theta}(t-\tau)\mathbf{B}\mathbf{F}(\tau)$. In the previous section, a procedure giving numerically the CF has been given through Eq. (13). Lastly, the response CF $M_{\mathbf{X}(t)}(\omega, t)$ can be obtained by considering an important property of the CF of the sum of two independent random processes (Soong, 1973), such that:

$$M_{\mathbf{X}(t)}(\omega, t) = M_{\bar{\mathbf{X}}_0(t)}(\omega, t) M_{\bar{\mathbf{X}}_F(t)}(\omega, t) \tag{17}$$

Even in this case, the response PDF is obtained by the Fourier anti-transform of the above CF.

## 4. Numerical example

Consider an $n$-degree of freedom structure, subject to a time-dependent vector-force $\mathbf{F}(t)$. The equation of motion is:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) &= \mathbf{F}(t) \\ \mathbf{U}(0) &= \mathbf{U}_0 \end{aligned} \tag{18}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, respectively; $\mathbf{U}(t)$ is the $n$-vector of the response displacements, $\mathbf{U}_0$ defines the initial conditions of the response and $\mathbf{F}(t)$ represents the excitation vector.

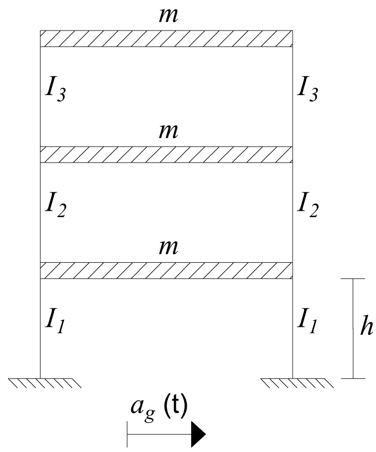


*Figure 2.*: Shear-type plane system.

By introducing the state variables vector $\mathbf{X}^T(t) = \left(\mathbf{U}^T(t)\ \ \dot{\mathbf{U}}^T(t)\right)$, Eq. (18) can be converted into the following first-order differential system:

$$\begin{aligned} \dot{\mathbf{X}}(t) &= \mathbf{D}\mathbf{X}(t) + \mathbf{v}\mathbf{F}(t); \\ \mathbf{X}(0) &= \mathbf{X}_0 \end{aligned} \tag{19}$$

where:

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix}; \quad \mathbf{v} = \begin{pmatrix} \mathbf{0} \\ -\mathbf{M}^{-1} \end{pmatrix}; \quad \mathbf{X}_0 = \begin{pmatrix} \mathbf{U}(0) \\ \dot{\mathbf{U}}(0) \end{pmatrix} \tag{20}$$

The response $\mathbf{X}(t)$ is evaluated taking into account the Duhamel integral, that is:

$$\mathbf{X}(t) = \mathbf{\Theta}(t)\mathbf{X}_0 + \int_0^t \mathbf{\Theta}(t-\tau)\mathbf{v}\mathbf{F}(\tau)d\tau \tag{21}$$

$\mathbf{\Theta}(t)$ being the transition (or fundamental) matrix corresponding to the dynamical system; it is expressed as:

$$\mathbf{\Theta}(t) = \exp(\mathbf{D}t) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{D}^n t^n. \tag{22}$$

Now, the shear-type plane system represented in Figure 2 is considered. The Young's modulus value is $E = 31 \times 10^9\ N/m^2$; all the columns have the same length $h = 3.2\ m$, while the moments of inertia of each column are $I_1 = I_2 = I_3 = 0.0054\ m^4$; for each floor, a mass $m = 50,000/g\ kg$ is assumed. The system is forced by a the zero-mean Gaussian stationary ground acceleration $a_g(t)$ defined by its one-side power Clough–Penzien spectra density having the expression:

$$S_a(\omega) = \frac{\omega_r^4 + 4\xi_r^2\omega_r^2\omega^2}{\left(\omega_r^2 - \omega^2\right)^2 + 4\xi_r^2\omega_r^2\omega^2} \frac{\omega^4}{\left(\omega_p^2 - \omega^2\right)^2 + 4\xi_p^2\omega_p^2\omega^2} \frac{0.141\xi_r^2 a_{g0}^2}{\omega_r\sqrt{1 + r\xi_r^2}} \tag{23}$$

where the following filtering coefficients have been considered: $\omega_p = 2.0$, $\omega_r = 19$, $\xi_p = 0.6$ and $a_{g0} = 0.25g$. Therefore, the vector of the external excitations $\mathbf{F}(t)$ of Eq. $(19_{(1)})$ is expressed as $\mathbf{M}\boldsymbol{\tau} a_g(t)$, where $\boldsymbol{\tau}$ is the structural incidence vector. While, the vector of the initial displacement condition in Eq. $(19_{(2)})$ , $\mathbf{X}_0$, is assumed as a random vector described by random variables uniformly distributed with $\sigma_{x_0} = 0.15$.

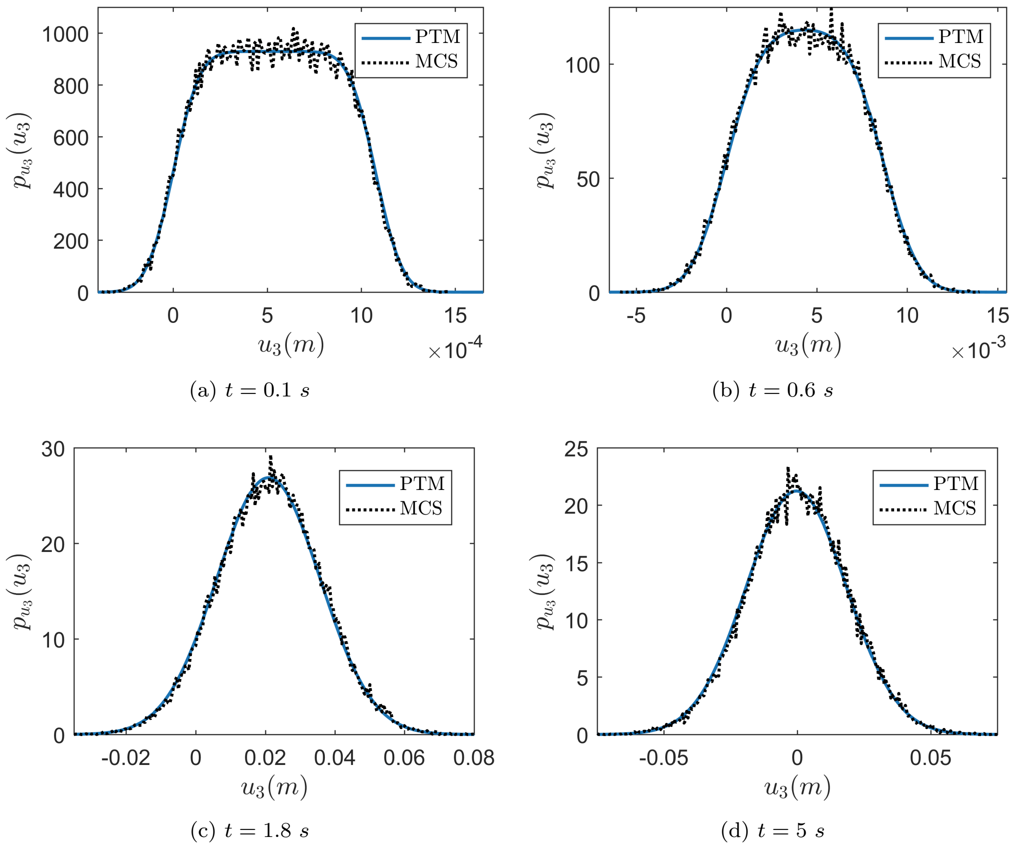


*Figure 3.*: Displacement PDF evaluated for four different instants. PTM (continuous line); MCS (dashed line)

The probabilistic definition of the displacement in correspondence of the $j$-th DOF at a fixed time $t_k$, that is $X^{(j)}(t_k)$, has been obtained by the application of EPTM. Figure 3, shows the PDF of the displacement $u_3$, in correspondence of 4 time instants. From the inspection of this figure, it is possible to appreciate that for the first instants, the random characteristics of the PDF outputs are significantly influenced by the random initial conditions, then the responses PDF are gradually filtered and depend only on the random characteristics of the excitation. These results are considered together with those coming out from an MCS characterized by $50,000$ samples. Overall, the analysis of the results evidences the goodness of the proposed stochastic procedure.

## 5. Conclusions

In this work, a stochastic procedure for the dynamic analyses of systems characterized by uncertainties in the external actions and in their initial conditions has been presented. Based on PTM, the EPTM combines the main properties of the mean square random calculus with the principle of conservation of probability. The proposed procedure allows to preserve the system's probability in time step by step and gives the random response directly in terms of CFs; then the response PDF of the system can be evaluated easily by the inverse Fourier transform. The application of the EPTM for the stochastic dynamic analysis of a shear-type system has been confirmed the goodness of the results in terms of evolutive PDF.

## References

Adhikari, S. Uncertainty propagation in linear systems: an exact solution using random matrix theory. *48th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference*, p. 1957, 2007

Bucher, C.G. and Schueller, G.I. Non-Gaussian response of linear system. *Structural Dynamics*, 103–127, 1991

Calatayud, J., Cortés, J.C., and Jornet, M. On the approximation of the probability density function of the randomized heat equation. *arXiv preprint arXiv:1802.04188*, 2018

Calatayud, J., Cortés, J.C., and Jornet, M. The damped pendulum random differential equation: A comprehensive stochastic analysis via the computation of the probability density function. *Physica A*, **512**, 261–279, 2018

Conte, J.P. and Peng, B.F. An explicit closed-form solution for linear systems subjected to nonstationary random excitation. *Probabilistic Engineering Mechanics*, **11**(1), 37–50, 1996

Di Paola, M. and Falsone, G. Non-Linear Oscillators Under Parametric and External Poisson Pulses. *Journal of Nonlinear Dynamics*, **5**(3), 337–352, 1994

Di Paola, M. and Falsone, G. Higher Order Statistics of the Response of MDOF Linear Systems Excited by Linearly Parametric White Noises and External Excitations. *Probabilistic Engineering Mechanics*, **12**(3), 179–188, 1997

Di Paola, M. and Falsone, G. Higher Order Statistics of the Response of MDOF Systems under Polynomials of Filtered Normal White Noises. *Probabilistic Engineering Mechanics*, **12**(3), 189–196, 1997

Di Paola, M., Falsone, G. and Pirrotta, A. Stochastic Response Analysis of Nonlinear Systems under Gaussian Inputs. *Probabilistic Engineering Mechanics*, **7**(1), 15–21, 1992

Falsone, G. Cumulants and Correlations for Linear Systems under Non-Stationary Delta-Correlated Processes. *Probabilistic Engineering Mechanics*, **9**(3), 157–165, 1994

Falsone, G. An extension of the Kazakov relationship for non-Gaussian random variables and its use in the non-linear stochastic dynamics. *Probabilistic Engineering Mechanics*, **20**(1), 45–56, 2005

Falsone, G. and Laudani, R. A probability transformation method (PTM) for the dynamic stochastic response of structures with non-Gaussian excitations. *Engineering Computations*, **35**(5), 1978–1997, 2018

Falsone, G. and Laudani, R. Matching the principal deformation mode method with the probability transformation method for the analysis of uncertain systems. *Engineering Computation*, 118(7): 395–410, 2019.

Falsone, G. and Laudani, R. Multi-time probability density functions of the dynamic non-Gaussian response of structures. *Structural Engineering and Mechanics*, **76**(5): 631, 2020

Falsone, G. and Settineri, D. Explicit solutions for the response probability density function of linear systems subjected to random static loads. *Probabilistic Engineering Mechanics*, 33: 86–94, 2013a.

Falsone, G. and Settineri, D. Explicit solutions for the response probability density function of nonlinear transformations of static random inputs. *Probabilistic Engineering Mechanics*, 33: 79–85, 2013b.

Gioffrè, M. and Gusella, V. Numerical analysis of structural systems subjected to non-Gaussian random fields. *Meccanica*, **37**(1-2), 115–128, 2012

Hussein, A. and Selim, M.M. Limit analysis of stochastic structures in the framework of the Probability Density Evolution Method. *The European Physical Journal Plus*, **30**(12), 249, 2015

Kalogeris, I. and Papadopoulos, V. Solution of the stochastic generalized shallow-water wave equation using RVT technique. *Engineering Structures*, **160**, 304–313, 2018

Li, J. (2016), Probability density evolution method: Background, significance and recent developments. *Probabilistic Engineering Mechanics*, **44**, 111–117, 2016

Li, J. and Chen, J. Stochastic dynamics of structures. *John Wiley & Sons*, 2009

Lin, Y.K. Probabilistic theory of structural dynamics. *McGraw Hill*, New York, NY, USA, 1967

Liu, Z. and Liu, Z. Random function representation of stationary stochastic vector processes for probability density evolution analysis of wind-induced structures. *Mechanical Systems and Signal Processing*, **106**, 511–525, 2018

Lutes, L.D. and Sarkani, S. Random vibration analysis of structural and mechanical system. *Butterworth-Heinemann*, Oxford, United Kingdom, 2004

Mamis, K.I., Athanassoulis G.A. and Kapelonis Z.G. A systematic path to non-Markovian dynamics: new response probability density function evolution equations under Gaussian coloured noise excitation. *Proceedings of the Royal Society A*, **475**, 2226, 2019

Makarios, T. K. Evaluating the effective spectral seismic amplification factor on a probabilistic basis. *Structural Engineering and Mechanics*, **42**(1), 121–129, 2012

Mazelsky, B. Extension of power spectral methods of generalized harmonic analysis to determine non-Gaussian probability functions of random input disturbances and output responses linear systems. *Aeronaut*, **21**(3), 145–153, 1954

Meimaris, A.T., Kougioumtzoglou, I.A. and Pantelous A.A. A closed form approximation and error quantification for the response transition probability density function of a class of stochastic differential equation. *Probabilistic Engineering Mechanics*, **54**, 87–94, 2019

Morikawa, H. and Kameda, H. Stochastic interpolation of earthquake ground motions under spectral uncertainties. *Structural Engineering and Mechanics*, **5**(56), 839–851, 1997

Roberts, J.B. and Spanos, P.D. Random vibration and statistical linearization. *Wiley*, Chichester, NY, USA, 1991

Soong, TT. Random Differential Equations in Science and Engineering. *Academic Press, New York*, 1973

Wu, W.F. and Lin, Y.K. Cumulant-neglect closure for non-linear oscillators under random parametric and external excitations. *International Journal of Non- Linear Mechanics*, **15**(5), 910–916, 1984