A Fully Decoupled Approach for a Class of Reliability-based Optimization Problems in Stochastic Linear Dynamics

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Abstract. Reliability-based optimization (RBO) allows determining optimal structural designs while explicitly taking into account the effects of uncertainty on structural performance. Its practical implementation is quite challenging from a numerical viewpoint, as it demands solving simultaneously a reliability problem nested in an optimization procedure, that is, a double-loop problem. This contribution develops a most efficient approach for RBO that avoids the aforementioned double-loop implementation. This approach is applicable for a specific class of problems: the minimization of the failure probability of linear structural systems subject to Gaussian stochastic loading. The proposed approach is formulated within the framework of the operator norm theorem. In this way, the RBO problem is reduced to the solution of a single deterministic optimization problem followed by a single reliability analysis, avoiding a double-loop (or nested) implementation. The application and capabilities of the proposed approach are illustrated by means of an example, indicating that the involved numerical costs can be reduced drastically.

Keywords: Reliability-based optimization, Linear dynamics, First excursion probability, Gaussian load, Operator norm.

1. Introduction

Reliability-based optimization (RBO) is a methodology that can assist decision-making under uncertainty in structural mechanics. In fact, it provides the means for determining the best design solution with respect to a certain criterion while explicitly taking into account the effects of uncertainty (Schuëller and Jensen, 2008). The advantages of RBO over traditional deterministic approaches are evident. Nonetheless, its practical implementation is extremely challenging from a numerical viewpoint, as it demands assessing different design solutions and for each of them, it is necessary to perform uncertainty quantification. In other words, RBO corresponds to a so-called *double-loop problem*, where reliability analysis is embedded within an optimization algorithm. Therefore, different strategies for solving RBO problems have been developed to decrease numerical costs associated with its practical implementation. These strategies involve, for example, building surrogate models, application of stochastic search techniques, etc. For a detailed overview about these strategies, it is referred existing overview papers in the literature, such as e.g. (Aoues and Chateauneuf, 2010, Enevoldsen and Sørensen, 1994, Schuëller and Jensen, 2008, Valdebenito and Schuëller, 2010).

This paper develops a framework for solving RBO problems for a specific class of applications: minimization of the first excursion probability of linear structures subject to dynamic loading of the Gaussian type. The proposed framework fully exploits the particular characteristics of the class of applications considered and is formulated in a *fully decoupled* fashion. This means that the RBO problem is solved by conducting a single deterministic optimization procedure followed by a single reliability estimation. The theoretical basis for formulating this fully decoupled framework lies on the operator norm theorem (see, e.g. (Tropp, 2004)), which has already been explored by the authors within the context of imprecise reliability analysis (Faes et al., 2020, Faes et al., 2021).

2. Formulation of the Problem

2.1. Stochastic Loading

Consider a zero-mean dynamic Gaussian loading f acting over a structural system. This Gaussian process is represented at discrete time instants $t_k = (k-1)\Delta t,\ k = 1, \ldots, n_T$, where Δt is the time step; n_T is the total number of time instants considered and $T = (n_T - 1)\Delta t$ is the load duration. Applying the Karhunen-Loève expansion (see, e.g. (Schenk et al., 2005, Stefanou, 2009)), the Gaussian loading is described as:

$$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{\Psi}\boldsymbol{\Lambda}^{1/2}\boldsymbol{z} \tag{1}$$

where $\boldsymbol{f}$ is a realization of the stochastic loading, which is a $n_T \times 1$ vector, whose k-th component f_k represents the loading at time t_k; $\boldsymbol{\Psi}$ is a matrix of dimension $n_T \times n_{KL}$ containing the first n_{KL} eigenvectors of the covariance matrix $\boldsymbol{\Gamma}$ associated with the Gaussian process; $\boldsymbol{\Lambda}$ is a matrix whose diagonal contains the first n_{KL} eigenvalues of the covariance matrix $\boldsymbol{\Gamma}$; n_{KL} is the number of terms retained for the Karhunen-Loève expansion ($n_{KL} \leq n_T$, see, e.g. (Stefanou, 2009)); and $\boldsymbol{z}$ is a realization of a standard Gaussian random variable vector $\boldsymbol{Z}$ of dimension $n_{KL} \times 1$ and whose probability density function is denoted as $p_{\boldsymbol{Z}}(\boldsymbol{z})$.

2.2. Stochastic Response

The stochastic Gaussian loading described above acts over a linear elastic structure with classical damping. Some of the parameters of this structure (such as elements' cross sections) can be chosen by the designer and are termed as *design variables*. These design variables are collected in a vector $\boldsymbol{y}$ of dimension n_y and can alter the structural behavior of the system (for example, stiffer structural members may help in reducing displacements).

For practical design purposes, it is of interest monitoring n_R responses of interest of the structure. In view of linearity, these responses are calculated by means of a convolution integral. Taking into account the discrete time representation of the dynamic stochastic loading, the different responses of interest at different time instants can be calculated in discrete form as:

$$\boldsymbol{\eta}_i(\boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{A}_i(\boldsymbol{y})\boldsymbol{z},\ i = 1, \ldots, n_R \tag{2}$$

where $\boldsymbol{\eta}_i$ is a vector of dimension $n_T \times 1$ containing the discrete-time representation of the i-th response along the duration T of the stochastic loading; and $\boldsymbol{A}_i$ is a matrix of dimension $n_T \times n_{KL}$ whose k-th row contains the discrete time representation of the convolution integral (Faes et al., 2020).
The responses of interest should remain below acceptable threshold levels $b_i,\ i = 1, \ldots, n_R$ within the duration T of the stochastic load in order to avoid an undesirable behavior. For verifying such condition, it is useful to define the so-called normalized response function $r(\boldsymbol{y}, \boldsymbol{z})$ (Au and Beck, 2001), which is defined as:

$$r(\boldsymbol{y}, \boldsymbol{z}) = ||\boldsymbol{A}(\boldsymbol{y})\boldsymbol{z}||_\infty \tag{3}$$

where $||\cdot||_\infty$ denotes infinity norm; and $\boldsymbol{A}$ is a matrix that collects all matrices $\boldsymbol{A}_i$ associated with the calculation of the i-th response (see eq. (2)), normalized by their respective threshold levels. Matrix $\boldsymbol{A}$ is defined as:

$$\boldsymbol{A}(\boldsymbol{y}) = \begin{bmatrix} b_1^{-1}\boldsymbol{A}_1(\boldsymbol{y}) \\ \vdots \\ b_{n_R}^{-1}\boldsymbol{A}_{n_R}(\boldsymbol{y}) \end{bmatrix} \tag{4}$$

and its dimension is $(n_R n_T) \times n_{KL}$.

2.3. First Excursion Probability

The chance that any of the responses of interest exceed their prescribed threshold within the duration of the stochastic loading is quantified by means of the following classical probability integral:

$$p_F(\boldsymbol{y}) = \int_{\boldsymbol{z}\in\mathcal{R}^{n_{KL}}} I_F(\boldsymbol{y}, \boldsymbol{z}) p_{\boldsymbol{Z}}(\boldsymbol{z}) d\boldsymbol{z} \tag{5}$$

where p_F denotes the failure probability; and $I_F(\cdot,\cdot)$ is the indicator function, which is equal to one in case $r(\boldsymbol{y}, \boldsymbol{z}) \geq 1$ and zero, otherwise. As the dimensionality of the failure probability integral is usually quite large (in the order of hundreds or thousands) and the indicator function is known point-wise for specific values of the design variable vector $\boldsymbol{y}$ and the stochastic excitation $\boldsymbol{z}$, eq. (5) must be usually evaluated by means of simulation, as discussed in (Schuëller and Pradlwarter, 2007). However, this is challenging from a numerical viewpoint, as it demands performing repeated dynamic analyses.

2.4. Reliability-based optimization

Recall that design variables $\boldsymbol{y}$ may be chosen by the designer in order to alter the dynamic performance of the structural system. A possible means for selecting these design variables is formulating a Reliability-based Optimization (RBO) problem. For example, one could carry minimization of failure probability under a constraint related with available resources (see, e.g. (Jensen et al., 2020)), which leads to the following optimization problem.

$$\begin{aligned}
&\min_{\boldsymbol{y}} p_F(\boldsymbol{y}) \\
&\text{subject to} \\
&c_l(\boldsymbol{y}) \leq 0,\ l = 1, \ldots, n_C \\
&y_j^L \leq y_j \leq y_j^U,\ j = 1, \ldots, n_y
\end{aligned} \tag{6}$$

where $c_l(\cdot)$ represents a deterministic constraint; y_j is the j-th element of $\boldsymbol{y}$; n_y denotes the dimension of the design variable vector; and y_j^L and y_j^U are the lower and upper bounds associated with y_j.

The solution of the optimization design problem in eq. (6) presents two challenges. First, the calculation of the failure probability for a given value of $\boldsymbol{y}$ by means of simulation demands performing repeated structural analyses for different realizations of the stochastic loading vector $\boldsymbol{z}$. Second, the calculation of the failure probability is embedded within the optimization algorithm, leading to the so-called double-loop implementation. These issues may impose a huge numerical cost. As a remedy to this situation, the application of the operator norm theorem is discussed in the remaining part of this work.

3. Operator Norm Theorem and its Application for Reliability-based Optimization

A close examination of eq. (3) reveals that the system's normalized response can be explained as the effect of the stochastic loading (represented in terms of $\boldsymbol{z}$) which acts over the system (represented in terms of the matrix $\boldsymbol{A}(\boldsymbol{y})$). In other words, the stochastic loading $\boldsymbol{z}$ is *stretched* by the system's matrix $\boldsymbol{A}(\boldsymbol{y})$. A bound on the amount of stretching exerted by $\boldsymbol{A}(\boldsymbol{y})$ is given by the operator norm theorem, that is (Tropp, 2004):

$$||\boldsymbol{A}(\boldsymbol{y})||_{\infty,2} = \inf\{c(\boldsymbol{y}) \geq 0 : ||\boldsymbol{A}(\boldsymbol{y})\boldsymbol{z}||_\infty \leq c(\boldsymbol{y})\,||\boldsymbol{z}||_2\} \tag{7}$$

where the amount of stretching is calculated with respect to the Euclidean norm of $\boldsymbol{z}$, as this can be loosely interpreted as the energy content of the stochastic load (Faes et al., 2020, Faes et al., 2021). The expression above implies that the stochastic load vector can be *stretched* by matrix $\boldsymbol{A}$ by factor c at most. Such an idea is illustrated schematically in Figure 1, where it has been assumed for simplicity that $n_R = 1$ and $n_T = n_{KL} = 2$. It is observed that a given realization $\boldsymbol{z}^{(1)}$ associated with the stochastic load produces a dynamic response given by $\boldsymbol{A}(\boldsymbol{y})\boldsymbol{z}^{(1)}$. None of the components of that dynamic response may exceed the associated scalar $c\left|\left|\boldsymbol{z}^{(1)}\right|\right|$.

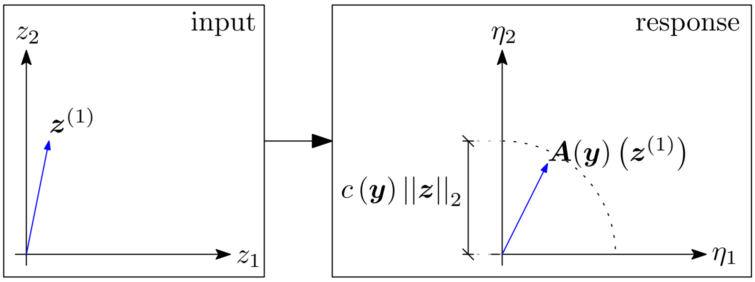

Figure 1. Schematic interpretation of operator norm theorem.

The solution to the operator norm problem in eq. (7) is equal to the row of matrix $\boldsymbol{A}$ with largest Euclidean norm, that is (Tropp, 2004):

$$||\boldsymbol{A}(\boldsymbol{y})||_{\infty,2} = \max_{i=1,\ldots,n_R,\ k=1,\ldots,n_T}\left(\sqrt{\boldsymbol{a}_{i_k}\boldsymbol{a}_{i_k}^T}\right) \tag{8}$$

where $\boldsymbol{a}_{i_k}$ represents the row of matrix $\boldsymbol{A}$ associated with the calculation of the i-th response at time instant t_k.

Note from the above discussion that the scalar c expresses the maximum amplification that the stochastic load may undergo when applied to the structure. As the normalized response function directly influences the calculation of the failure probability (see eq. (5)), this implies that minimizing the operator norm associated with $||\boldsymbol{A}(\boldsymbol{y})||_{\infty,2}$ offers a proxy which is equivalent to minimizing the failure probability. The quality of this proxy has been verified and validated, e.g. for estimating imprecise probabilities in (Faes et al., 2020, Faes et al., 2021). In view of this conclusion, the RBO problem formulated in eq. (6) can be replaced by the following problem:

$$\begin{aligned}
&\min_{\boldsymbol{y}} ||\boldsymbol{A}(\boldsymbol{y})||_{\infty,2}\\
&\text{subject to}\\
&c_l(\boldsymbol{y}) \leq 0,\ \ l = 1,\ldots,n_C\\
&y_j^L \leq y_j \leq y_j^U,\ \ j = 1,\ldots,n_y
\end{aligned} \tag{9}$$

This optimization problem in eq. (9) is actually deterministic, as it does not involve the stochastic load $\boldsymbol{z}$. Hence, once its optimum $\boldsymbol{y}^{\star}$ is found by means of any suitable optimization algorithm, it suffices to carry out a single reliability analysis for that optimum in order to determine the minimum failure probability $p_F\left(\boldsymbol{y}^{\star}\right)$ associated with the RBO problem in eq. (6).

As noted from the above description, the operator norm theorem framework offers the means for fully decoupling the solution of the RBO problem. That is, a nested problem is broken into a single deterministic optimization problem, followed by a single reliability problem. This is evidently most advantageous, as it decreases numerical costs substantially.

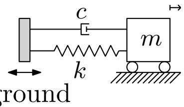

Figure 2. Single-degree-of-freedom (SDOF) oscillator subject to stochastic ground acceleration.

4. Test Example

Consider a single-degree-of-freedom (SDOF) oscillator with mass $m = 1$ [kg], nominal stiffness $k = 225$ [N/m] and classical damping $d = 5\%$, as depicted in Figure 2. The oscillator is subjected to a zero-mean stochastic Gaussian load of time duration $T = 20$ [s], discretized considering times steps of $\Delta t = 0.01$ [s]. This Gaussian load follows a modulated Clough-Penzien spectrum (see, e.g. (Zerva, 2009)) with: spectral intensity of 5×10^{-3} [kg m^2/s^3]; natural circular frequencies of 6π [rad/s] and 0.6π [rad/s] for the primary and secondary filters, respectively; damping ratios of 60%; and modulation function following the Shinozuka-Sato model with shape parameters $c_1 = 0.14$ and $c_2 = 0.16$ (Shinozuka and Sato, 1967).
The responses of interest are the maximum values of the relative displacement and absolute acceleration of the SDOF oscillator. The admissible threshold values for these two responses are 0.07 [m] and 7.5 [m/s^2]. The first excursion probability associated with these responses is calculated by means of Directional Importance Sampling (Misraji et al., 2020) considering a total of 1000 samples.
The objective of this example is determining the value of the stiffness y of the oscillator which minimizes its first excursion probability, under the constraint that y is a positive real value (that is, $y \in [0, \infty[$). Before solving the actual optimization problem, the failure probability is calculated over a discrete grid of values of the stiffness such that $y \in [70, 170]$ [N/m]. The estimates of both the failure probability and the associated operator norm for each of those discrete grid values is shown in Figure 3. It is noted that the minimum value of the failure probability is attained for a stiffness of about 105 [N/m]. The stiffness value for which the failure probability curve assumes its minimum matches with the value of the stiffness for which the minimum operator norm is attained, confirming the validity of applying the operator norm as a proxy for solving the RBO problem.

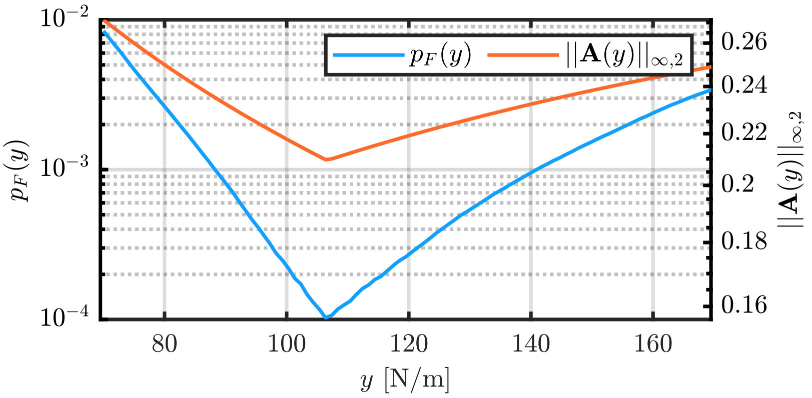

Figure 3. Failure probability and operator norm as a function of the stiffness y.

After these preliminary results, the actual RBO problem is solved using both a classical double-loop approach and the proposed, fully decoupled approach based on operator norm. The initial guess for the optimum is selected as $y^{(0)} = 90$ [N/m]. Then, the identified optimum results equal for both approaches (that is, double loop and proposed approach) and its numerical value is $y^* = 106.6$ [N/m]. However, the numerical costs are quite different. On one hand, for the double-loop implementation, the optimization solver demands a total of 52 function calls (that is, evaluations of the failure probability), each involving 1000 deterministic samples to compute the failure probability. In other words, a total of 52000 dynamic analyses are carried out. On the other hand, the solution by means of the proposed approach demanded 80 function calls (that is, calculation of the operator norm) followed by 1000 samples (dynamic analyses) for determining the reliability at the optimum. These results highlight the benefits of the proposed approach for RBO.

5. Conclusions

This paper has introduced an approach for solving a specific class of RBO problems. That is, minimization of the failure probability of linear systems subject to Gaussian stochastic loading. The approach is based on the operator norm and effectively decouples optimization from reliability assessment.
Although the results presented in this contribution are encouraging, it should be kept in mind that the proposed approach is applicable for a very specific types of problems. Therefore, the authors are working on the extension of this approach to nonlinear cases with the help of equivalent stochastic linearization.

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