Random set solutions to partial differential equations

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Engineering Models as Input-Output Systems

INPUT		OUTPUT	
parameters p	functions $q(\cdot)$	function $u(p, q(\cdot))$	values $h(u)$
random variables $p(\omega)$	random fields $q(\cdot, \omega)$	random function $u(p(\omega), q(\cdot, \omega))$	random variable $h(u(\omega))$
sets of random variables $p(\omega, I)$	sets of random fields $q(\cdot, \omega, I)$	set of random functions $u(p(\omega, I), q(\cdot, \omega, I))$	random interval $h(u(\omega, I))$

Want to compute: Probabilities, upper and lower probabilities, etc.

$$P\big(h(u(\cdot)) \in B\big),$$

$$\overline{P}(B) = P(h(u(\cdot, I)) \cap B \neq \emptyset), \quad \underline{P}(B) = P(h(u(\cdot, I)) \subset B)$$

Can we do that? For what B's? For which q's? For what u's?

Yes we can, in all reasonable cases. <u>This is what the talk is about.</u>

Random variables, informally: A random variable a is a random quantity, whose values are singletons in a target space $\mathcal{A}$.

Probabilities of an event $B \subset \mathcal{A}$: $P(B) = P(a \in B)$

Random variables, rigorously: Underlying probability space

$$(\Omega, \Sigma, P).$$

Σ is a collection of subsets of Ω, the measurable subsets, to which a probability can be assigned: $P : \Sigma \to [0, 1]$.

What are the events B? They should include all open and all closed subsets of the target space $\mathcal{A}$ (more generally, all Borel sets).

A random variable is a map $a : \Omega \to \mathcal{A}$ such that all inverse images

$$a^{-1}(B) = \{\omega \in \Omega : a(\omega) \in B\}$$

belong to Σ (are measurable). Then one can assign

$$P(B) := P(a^{-1}(B)) = P(\omega \in \Omega : a(\omega) \in B)$$

Random Sets

Random sets, informally: A random set A is a random quantity, whose values are subsets of a target space $\mathcal{A}$.

Upper and lower probabilities of an event B:

$$\overline{P}(B) = P(A \cap B \neq \emptyset), \quad \underline{P}(B) = P(A \subset B)$$

Random sets, rigorously: Upper/lower inverses of events $B \subset \mathcal{A}$:

$$A^{-}(B) = \{\omega \in \Omega : A(\omega) \cap B \neq \emptyset\}, \quad A_{-}(B) = \{\omega \in \Omega : A(\omega) \subset B\}.$$

The requirements that $\omega \to A(\omega)$ is a <u>random set</u> are

- each $A(\omega)$ is closed;
- the upper inverses $A^{-}(B)$ are measurable subsets of Ω for every Borel subset B of the target space $\mathcal{A}$.

Then it is legitimate to introduce upper and lower probabilities

$$\begin{aligned}
\overline{P}(B) &= P\big(A^{-}(B)\big) = P(\{\omega \in \Omega : A(\omega) \cap B \neq \emptyset\}), \\
\underline{P}(B) &= P\big(A_{-}(B)\big) = P(\{\omega \in \Omega : A(\omega) \subset B\}).
\end{aligned}$$

Given a **family of random variables** with parameter range Λ, defined on a common probability space (Ω, Σ, P),

$$\omega \to a_\lambda(\omega), \quad \lambda \in \Lambda,$$

the **corresponding random set** is

$$\omega \to A(\omega) = \{a_\lambda(\omega) : \lambda \in \Lambda\}.$$

Example: Imprecise Gaussian family, $\omega \sim \mathcal{U}(0,1)$,

$$a_\lambda(\omega) = \mu + \sigma\Phi^{-1}(\omega) \sim \mathcal{N}(\mu, \sigma^2)$$

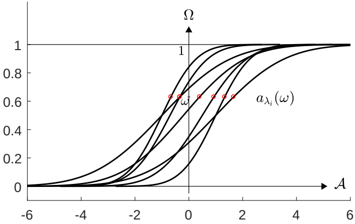
with interval parameters

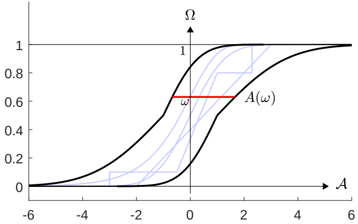
$$\lambda = (\mu, \sigma) \in \Lambda = [\underline{\mu}, \overline{\mu}] \times [\underline{\sigma}, \overline{\sigma}].$$

The continuous dependence of $a_\lambda(\omega)$ on λ implies that

$$A(\omega) = [\underline{a}(\omega), \overline{a}(\omega)] \quad \text{with}$$

$$\underline{a}(\omega) = \min\big(a_\lambda(\omega) : \lambda \in \Lambda\big), \quad \overline{a}(\omega) = \max\big(a_\lambda(\omega) : \lambda \in \Lambda\big).$$





Some members of the family a_λ (left), random set with focal element (right).

The upper and lower distribution functions define a p-box through

$$\overline{F}(b) = P(\omega : \underline{a}(\omega) \leq b), \quad \underline{F}(b) = P(\omega : \overline{a}(\omega) \leq b).$$

In the random interval case $[\underline{a}(\omega), \overline{a}(\omega)]$, simply:

- $\overline{F}$ is the distribution function of the random variable $\underline{a}$,
- $\underline{F}$ is the distribution function of the random variable $\overline{a}$.

The General Measurability Result

Hypotheses:

(A) (Ω, Σ, P) is a complete probability space;

(B) the target space $\mathcal{A}$ is a normed, complete and separable space;

(C) Λ is a bounded and closed subset of some Euclidean space;

(D) $a_\lambda(\omega)$, $\lambda \in \Lambda$ is a family of random variables with values in $\mathcal{A}$, that is, the maps $\omega \to a_\lambda(\omega)$ are measurable for each λ;

(E) the random variables depend continuously on λ, that is, the maps $\Lambda \to \mathcal{A} : \lambda \to a_\lambda(\omega)$ are continuous for each ω.

Conclusions:

- The set-valued map $\omega \to A(\omega) = \{a_\lambda(\omega) : \lambda \in \Lambda\}$ is a random set.
- All focal elements $A(\omega)$ are bounded and closed subsets of $\mathcal{A}$.
- In addition, if Λ is a (multi-dimensional) interval and $\mathcal{A} = \mathbb{R}$, the sets $A(\omega)$ are intervals.

(1) By continuity (e), all $A(\omega)$ are bounded and closed subsets of $\mathcal{A}$.

(2) Take a countable dense subset of parameter values λ_k in Λ, for example, all points with rational coordinates. By continuity (e), the set $\{a_{\lambda_k}(\omega) : k = 1, 2, 3, \ldots\}$ is dense in $A(\omega)$ for every fixed ω.

(3) Let B be an open set. Then

$$\begin{aligned} A^-(B) &= \{\omega \in \Omega : A(\omega) \cap B \neq \emptyset\} \\ &= \{\omega \in \Omega : \text{ there is } k \text{ such that } a_{\lambda_k}(\omega) \in B\} \\ &= \bigcup_k \{\omega \in \Omega : a_{\lambda_k}(\omega) \in B\} \end{aligned}$$

is measurable as a countable union of measurable sets (d).

(4) Using (a), (b), (c), the fundamental measurability theorem implies that $A^-(B)$ is measurable for every Borel set B.

This is the required property for $\omega \to A(\omega)$ to be a random set.

Parametrized Random Fields

Example: a mean zero random field $q(x, \omega, \ell)$, defined for x in a domain D, on a common probability space Ω with fixed variance σ^2, but correlation length ℓ varying in an interval $L = [\underline{\ell}, \overline{\ell}]$. Thus

- For fixed $x \in D$, $\ell \in L$, the map $\omega \to q(x, \omega, \ell)$ is measurable.

Additional assumptions:

- For fixed $x \in D$, $\omega \in \Omega$, the map $\ell \to q(x, \omega, \ell)$ is continuous.
- For fixed $\ell \in L$, $\omega \in \Omega$, the map $x \to q(x, \omega, \ell)$ is continuous.

The requirements are fulfilled, e.g., for random fields with autocorrelation functions of the form $c(\rho) = \exp(-\rho/\ell)$.

One can (and should) view $q(x, \omega, \ell)$ as a random variable with values in $\mathcal{C}(D)$, the space of continuous functions on D. Properties:

- for fixed $\ell \in L$, the map $\Omega \to \mathcal{C}(D)$, $\omega \to q(\cdot, \omega, \ell)$ is measurable;
- for fixed $\omega \in \Omega$, the map $L \to \mathcal{C}(D)$, $\ell \to q(\cdot, \omega, \ell)$ is continuous.

Model problem from elastostatics:

$$\begin{aligned} -\operatorname{div}\left(a(x)\operatorname{grad}u(x)\right) &= f(x), & x &\text{ in } D \\ u(x) &= 0, & x &\text{ in } \partial D \end{aligned}$$

where D is a bounded open domain in $\mathbb{R}^d$ and ∂D is its boundary.

Variational formulation:

Find $u \in H_0^1(D)$ such that, for all $v \in H_0^1(D)$,

$$\int_D a(x)\operatorname{grad}u(x)\cdot\operatorname{grad}v(x)\mathrm{d}x + \int_D f(x)v(x)\mathrm{d}x = 0$$

Deterministic case: $f \in L^2(D)$, $a \in \mathcal{C}(D;\alpha,\beta)$, i.e. $\alpha \le a(x) \le \beta$.
Then there is a unique solution $u = u_a \in H_0^1(D)$, and the map

$$a \in \mathcal{C}(D;\alpha,\beta) \to H_0^1(D) : a \to u_a$$

is continuous.

Taking the coefficient a as parametrized random field:

$$a(x,\omega,\ell) = \mu(x) + \chi\big(q(x,\omega,\ell)\big), \quad \text{for} \quad x = \ \text{in } D, \ell \in L.$$

with deterministic mean field μ and cut-off χ (ensuring bounds α, β). Viewing $a = a(\cdot,\omega,\ell)$ as a random variable with values in $\mathcal{C}(D;\alpha,\beta)$, previous continuity assertions imply

- for fixed $\ell \in L$, the map $\Omega \to H_0^1(D)$, $\omega \to u_{a(\cdot,\omega,\ell)}$ is measurable;
- for fixed $\omega \in \Omega$, the map $L \to H_0^1(D)$, $\ell \to u_{a(\cdot,\omega,\ell)}$ is continuous.

By the general measurability result, the collection of solutions

$$U(\omega) = \{u_{a(\cdot,\omega,\ell)} : \ell \in L\}$$

is a random set in the target space $\mathcal{A} = H_0^1(D)$.
Further, the evaluations at (almost all) points $x \in D$,

$$U(\omega, x) = \{u_{a(\cdot,\omega,\ell)}(x) : \ell \in L\}$$

are random intervals in the target space $\mathcal{A} = \mathbb{R}$.

Result

Finally, the Aumann expectations

$$U(x) = \mathrm{E}(U(\cdot, x))$$

form an interval field with domain D.

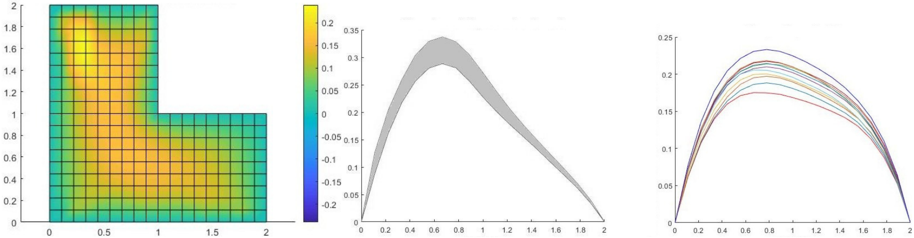


FIGURE: Aspects of the solution random set U (with fixed load $f(x) \equiv 1$). Trajectory of displacement random field at fixed correlation length $\ell = 0.5$ (left); slice at $x_2 = 0.4444$ through a realization of the full random set solution (center); slice at $x_2 = 0.4444$ through the mean value interval field (right), showing individual mean fields for varying correlation lengths inside $(0.5 \leq \ell \leq 1.5)$.