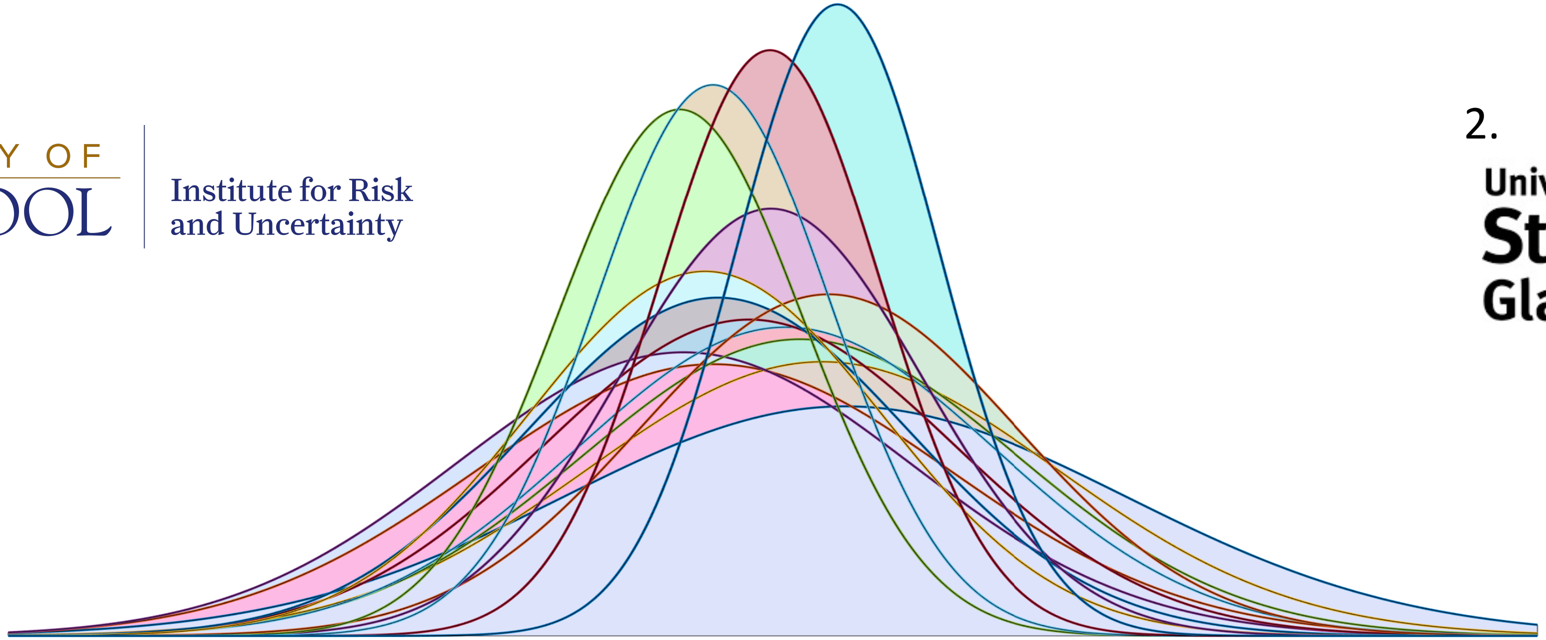
What's $Z - X$, when $Z = X + Y$?

Dependency tracking in interval arithmetic

1.

Institute for Risk and Uncertainty

2.



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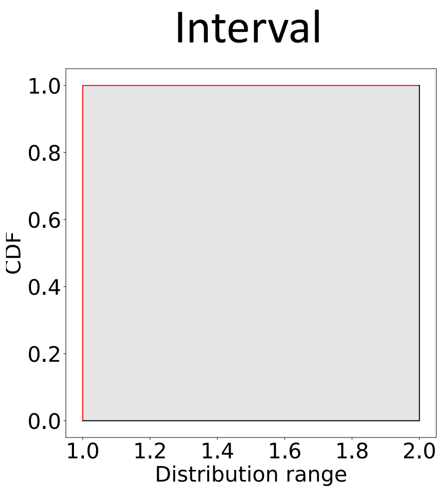
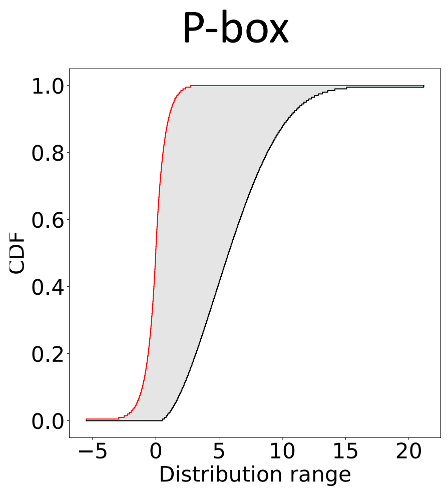
REC2021

Uncertainty Arithmetic





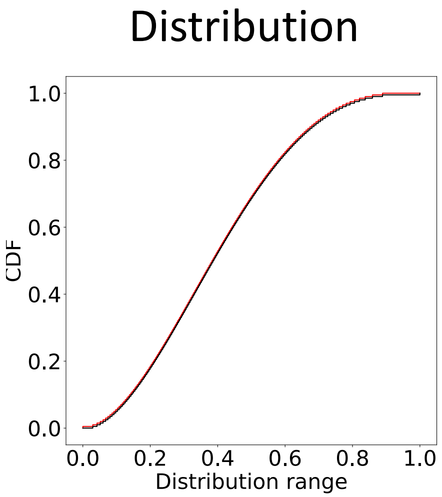
- Give uncertain extensions to deterministic function f

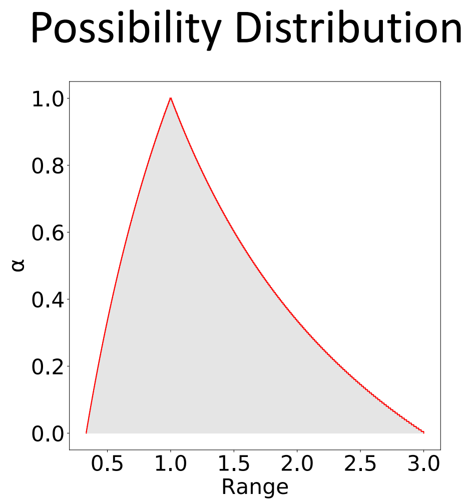
$$f: R^n \rightarrow R^m$$

- Replace floating point operations with uncertain operations (+, -, ×, ÷, ^, min, max, exp, sin, cos, …)

- Give outer approximations to uncertainty

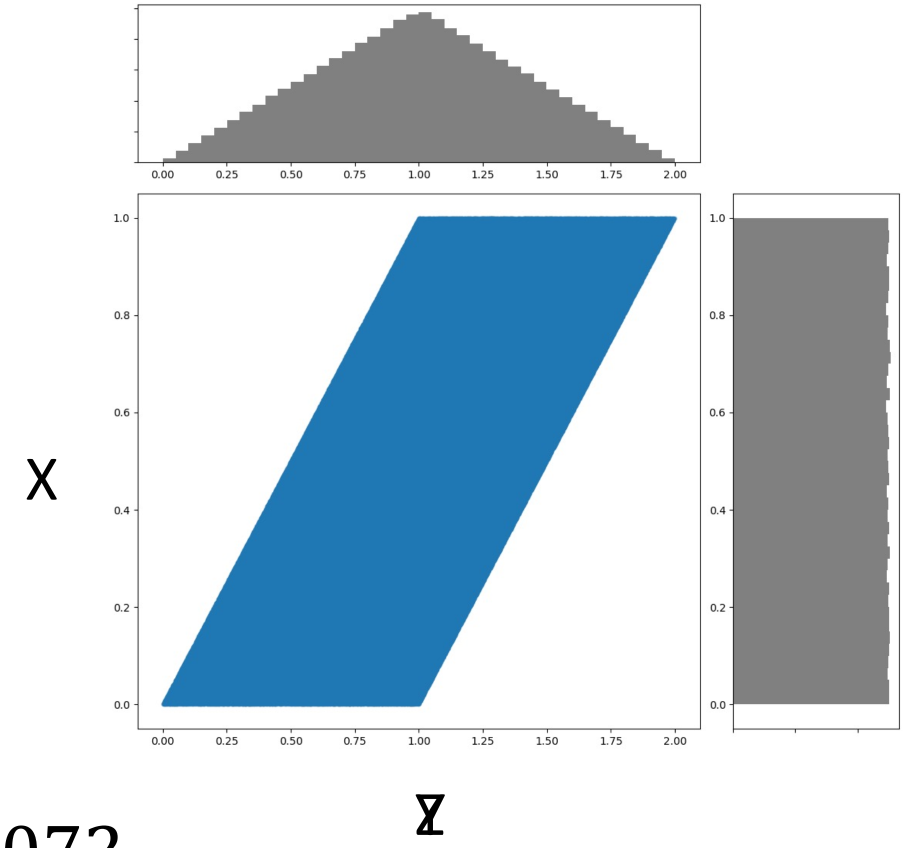
- Arithmetic without any knowledge about dependencies (Fréchet)





$$Z = X + Y \quad \Big| \quad [-1, 1] + [-1, 1] = [-2, 2]$$

$$Y \subset \tilde{Y} \qquad [-1, 1] \subset [-3, 3]$$



Even if *X* and *Y* are non-dependent,
Z and *X* are dependent

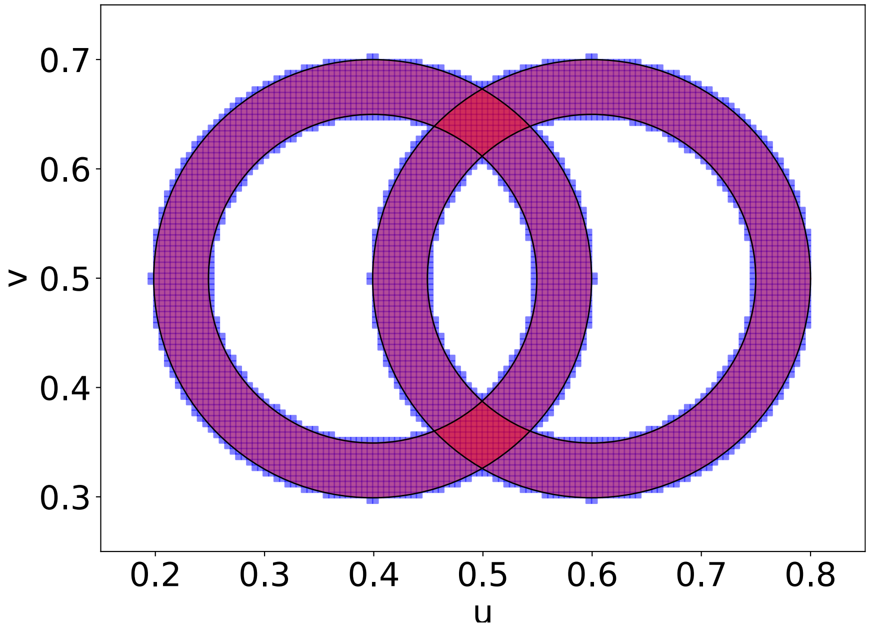
$$\rho_{ZX} = 0.7072$$

Interval Dependencies

Copula: $C: [0,1]^2 \rightarrow [0,1]$

Interval Dependence: $\mathbb{Z}: [0,1]^2 \rightarrow \{0,1\}$

$$Z(u,v) = \begin{cases} 1 & \text{if } (u \times v) \cap \mathbb{Z} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

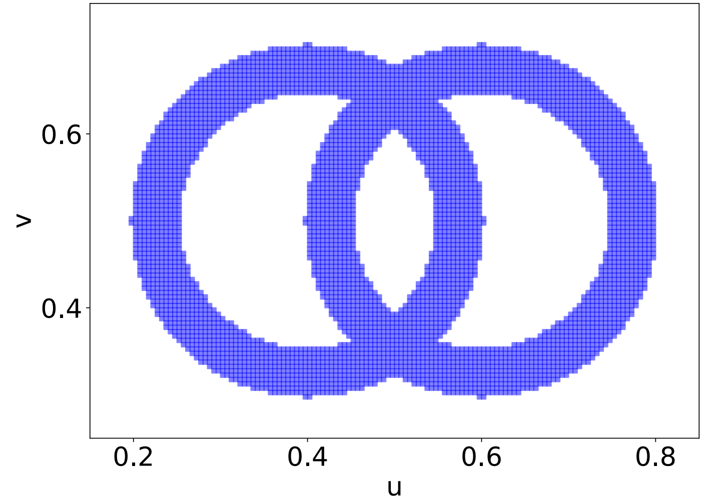


“Zones”

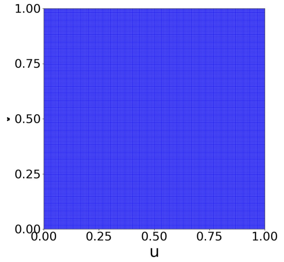
- Returns 0 if an interval can be excluded with 100% certainty
- Don’t store each subinterval
- N x N Boolean matrix
- The interval analogue to a copula

$$J_{XY}(x,y) = supp\ Z(u,v)$$

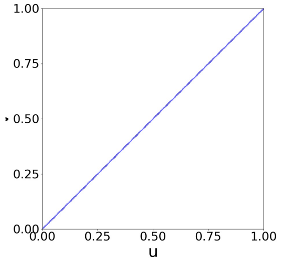
$$u = (x - \underline{X})/(\overline{X} - \underline{X}) \qquad v = (y - \underline{Y})/(\overline{Y} - \underline{Y})$$



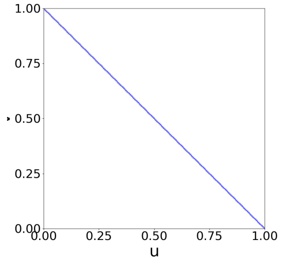
Non-interactive



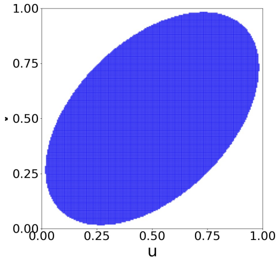
Perfect



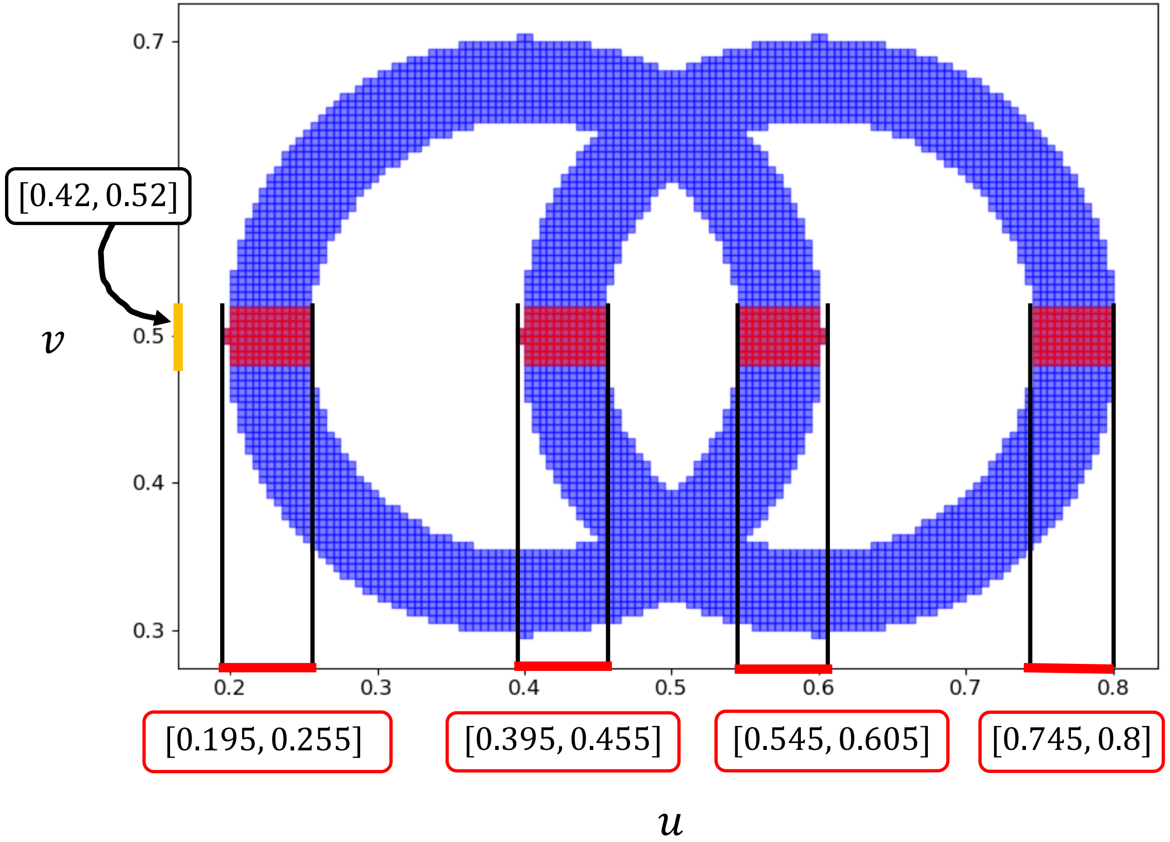
Opposite



Ellipse



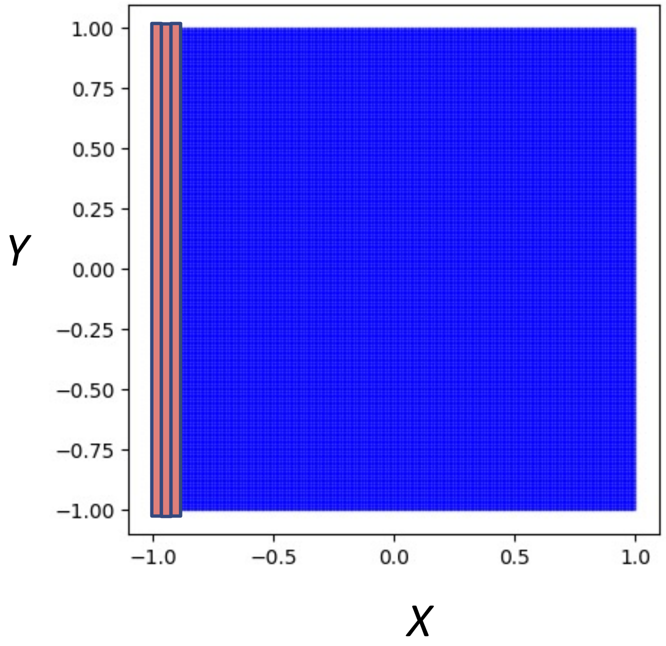
Conditioning



$$J(x|\, y = Y) = J(x, y) \cap ([-\infty, \infty] \times Y)$$

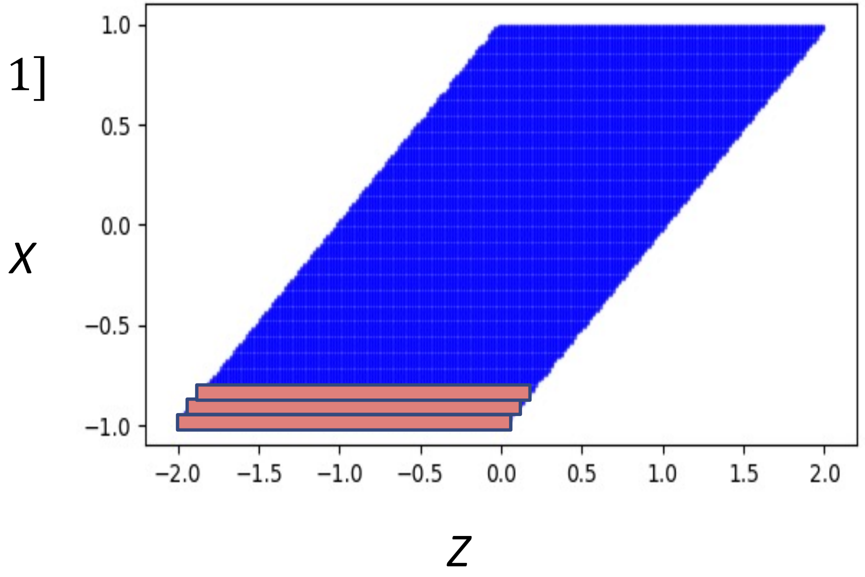
Zone Arithmetic

$$Z = X + Y$$



$$Z_{X1} = [-1, -0.99] + [-1, 1]$$

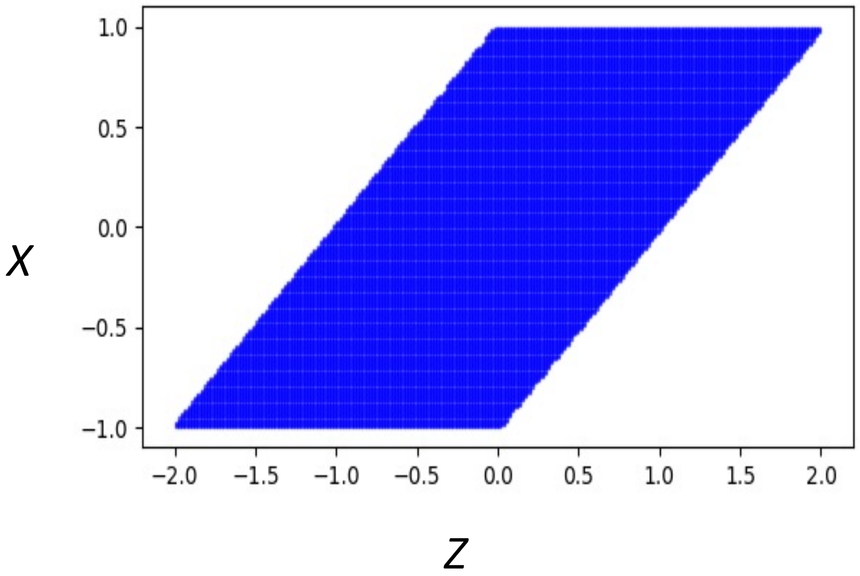
$$J_{Z,X_1} = Z \cap Z_{X1}$$

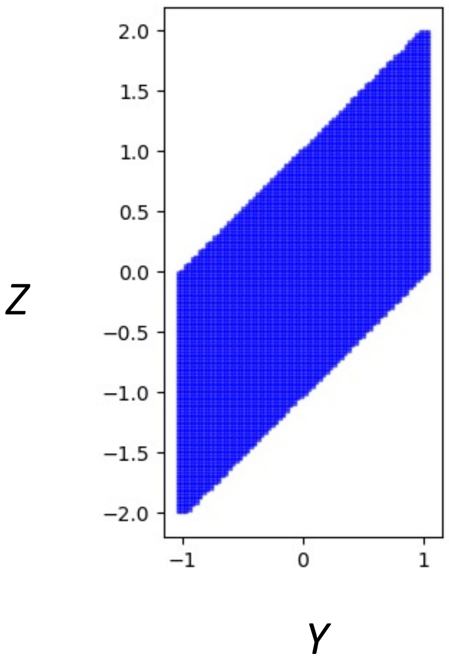


Z - X

$$Z = [-1, 1] + [-1, 1] = [-2, 2]$$

$$Z - X = [-2, 2] - [-1, 1] = [-3, 3] \quad \neq Y$$





$$Z - X = [-1.1, 1.1] \approx Y$$

Repeated Operations

$$X = [-1, 1]$$

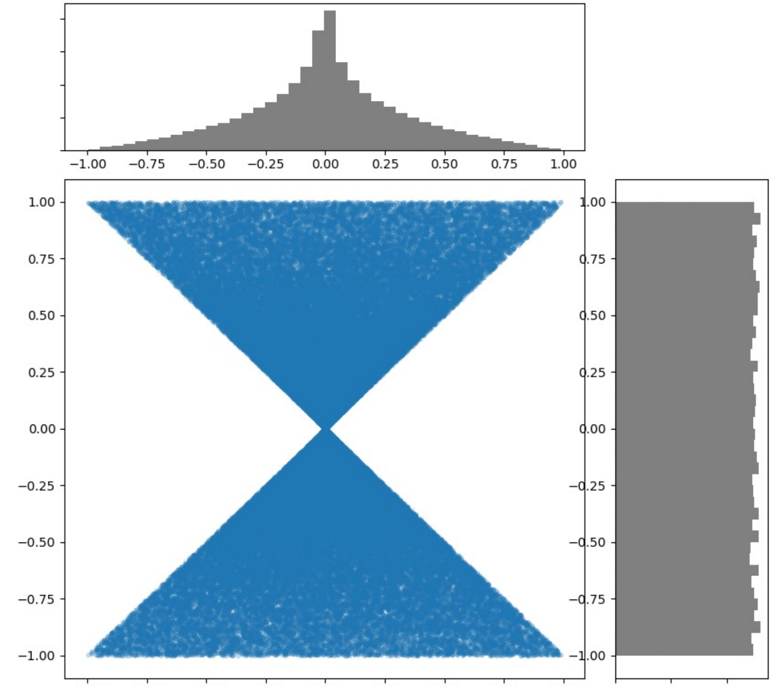
$$Y = [-1, 1]$$

$$Z = X * Y$$

$$\begin{aligned} &for\ i = 1{:}12 \\ &\qquad Z = Z * X \\ &end \end{aligned}$$

Monte Carlo

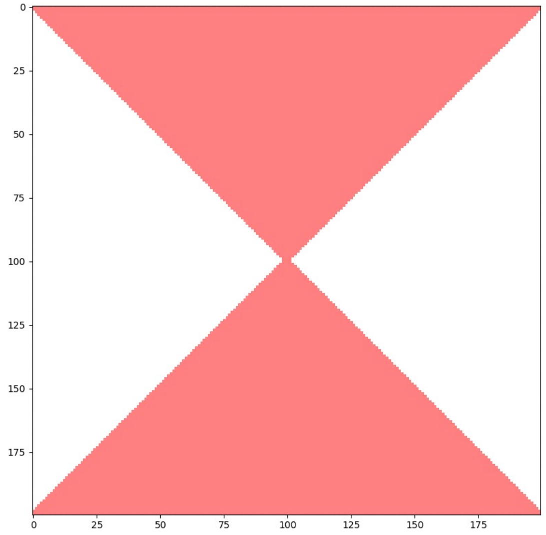
X



Z

Zone Arithmetic

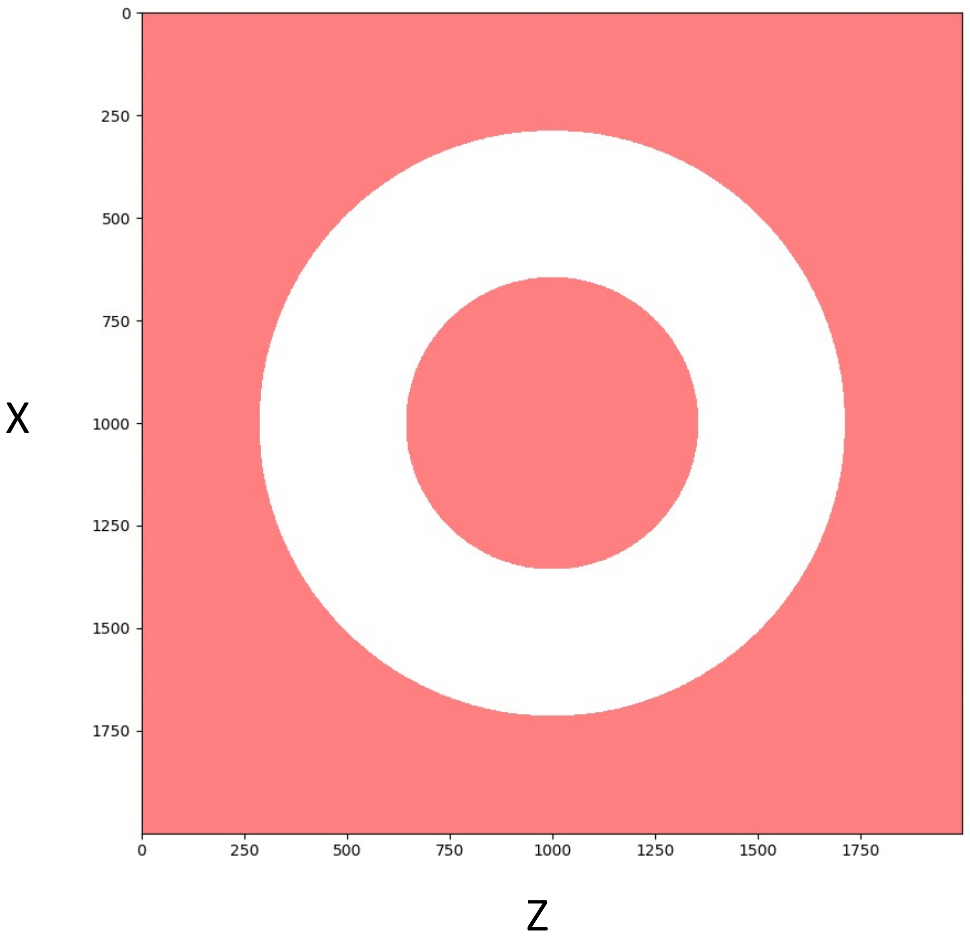
X



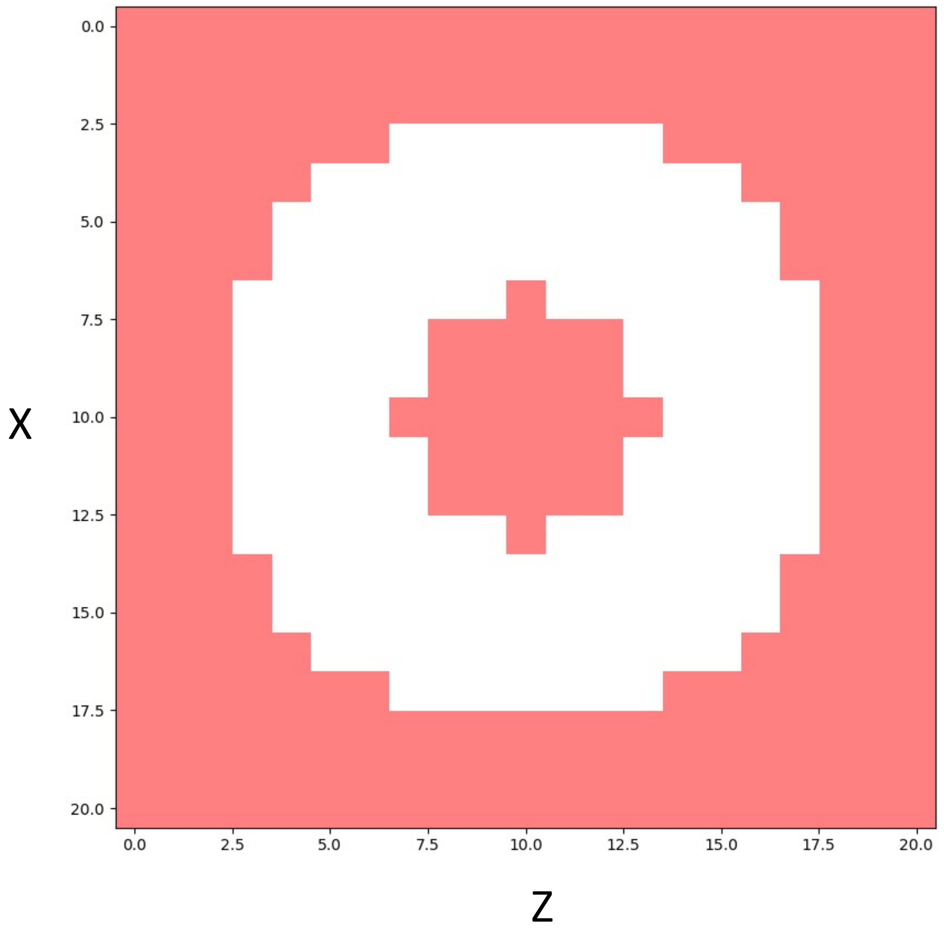
Z

Conservatism

2001 x 2001



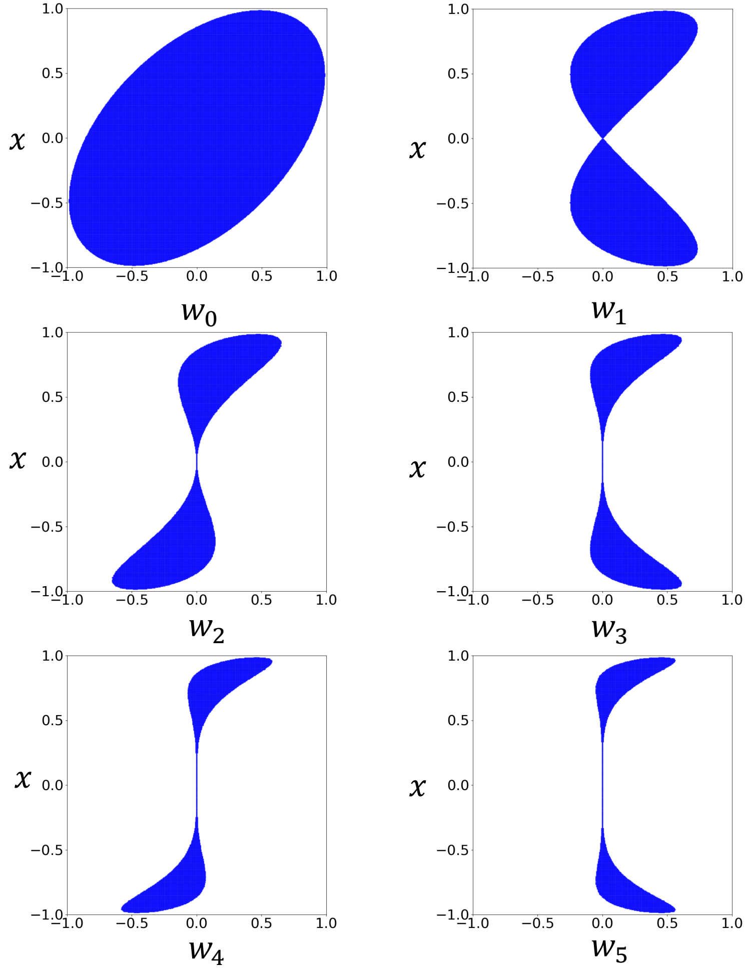
21 x 21



ZoneArithmetic.jl

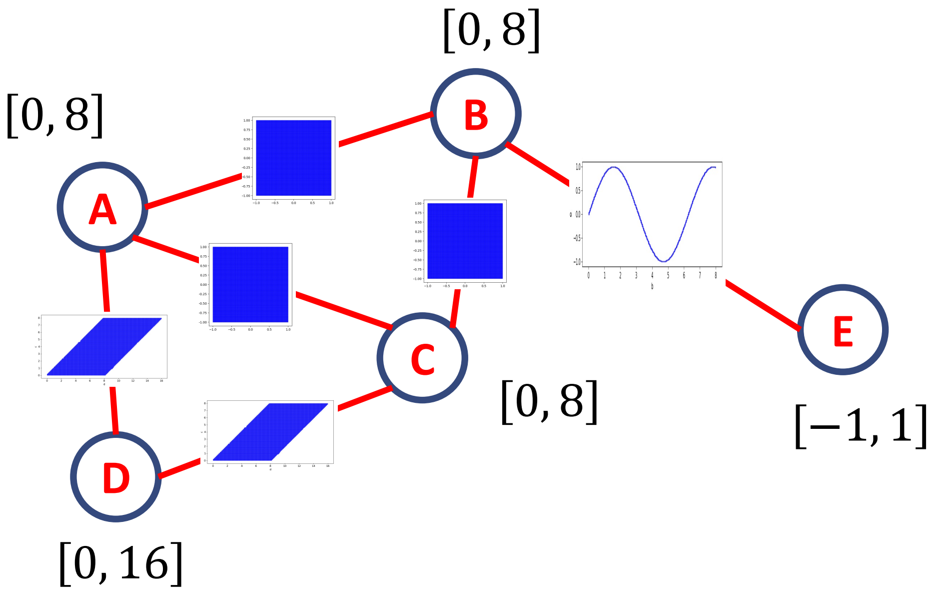
https://github.com/AnderGray/ZoneArithmetic.jl

```julia
using IntervalArithmetic, ZoneArithmetic

x = intervalZ(-1, 1)
w = intervalZ(-1, 1)

zone = ellipse(e = 0.5, n = 2000)
add_dep(w, x, zone)

for i = 1:5
    w = w * x
end

Jwx = joint(w, x)
plot(w, x)
```



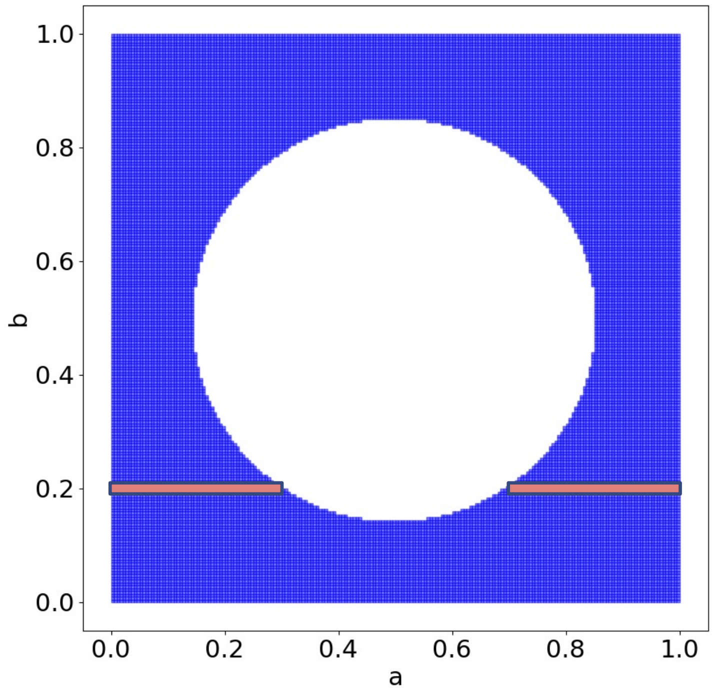
Tracking dependencies

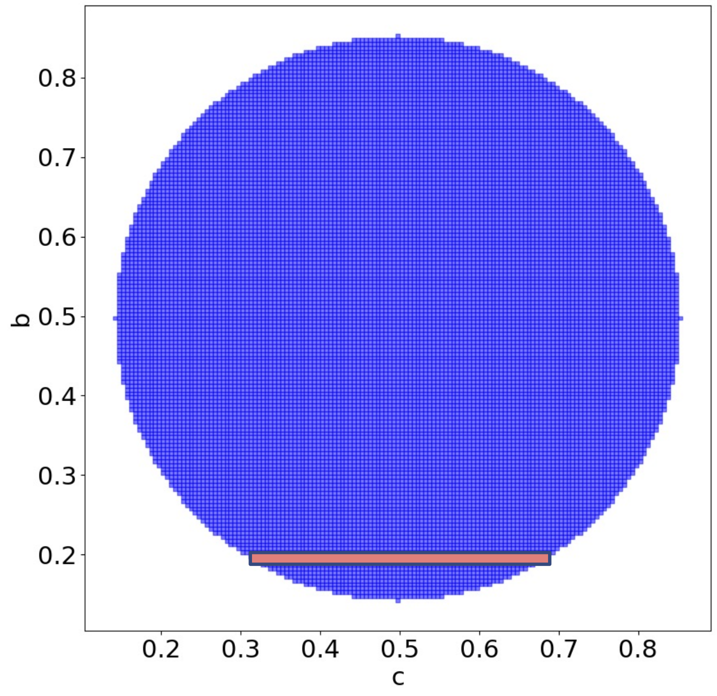
- Dependencies can be tracked in a graph
- The nodes are variables (intervals)
- The edges are dependencies (zones)
- When a new variable is calculated, it's added and linked in the graph

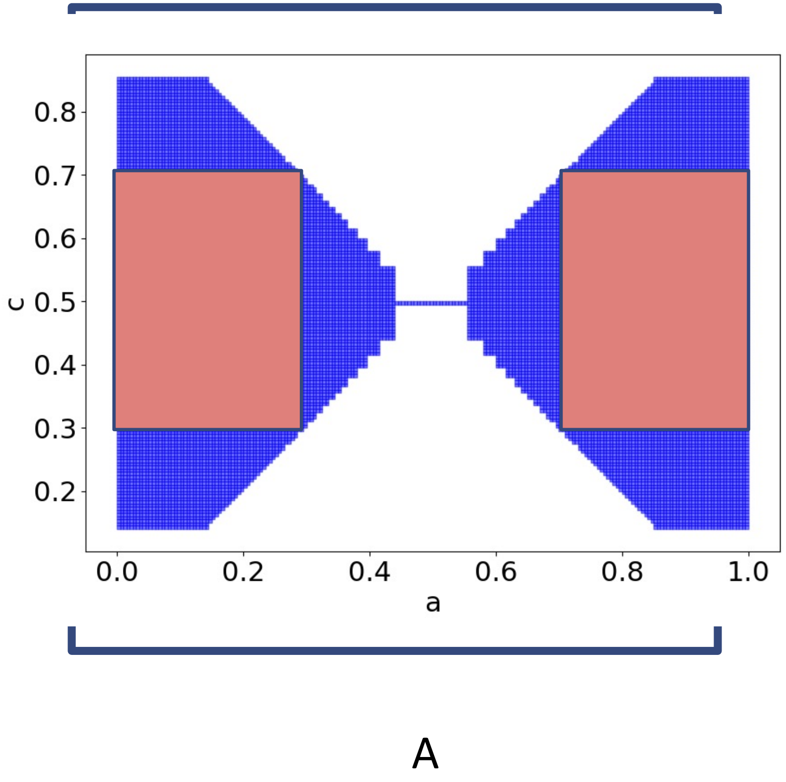



```
julia> d = a + c
intervalZ([0, 16], 4)
```

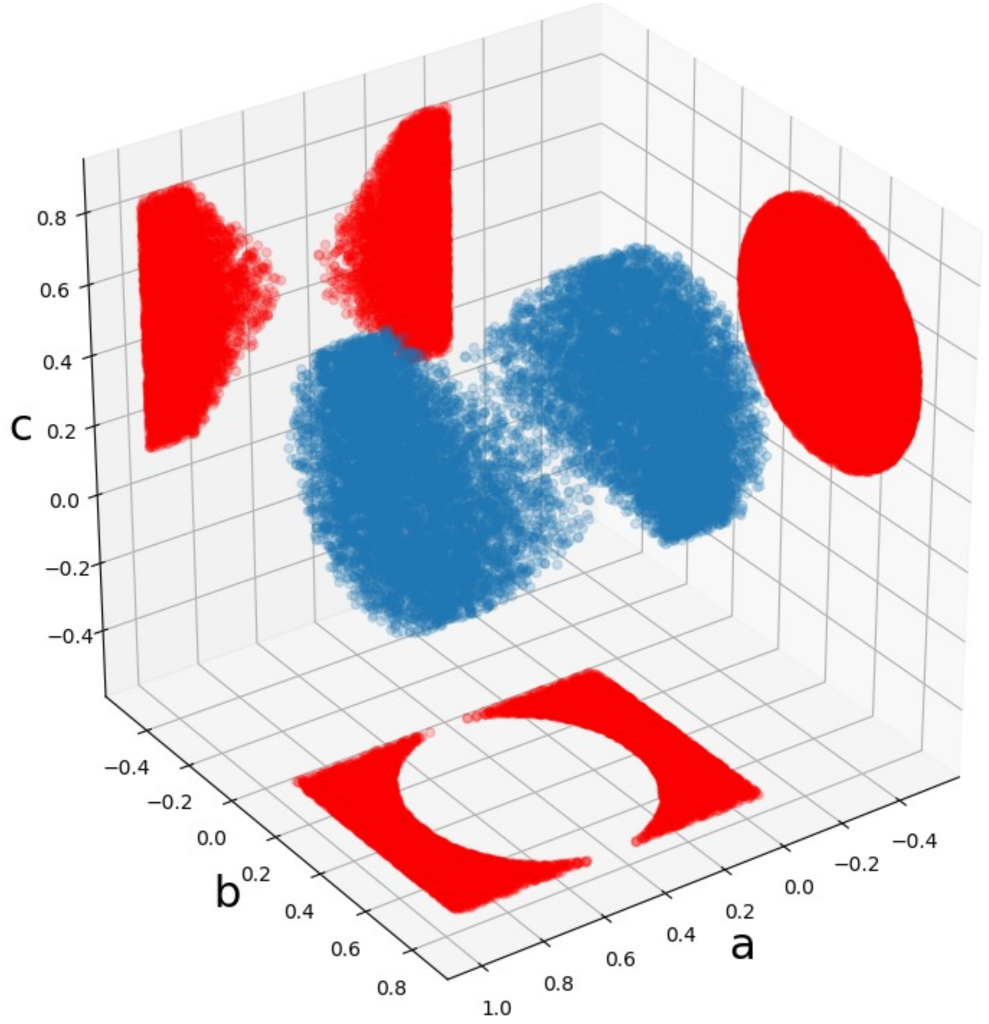
Zone projection







Zone projection

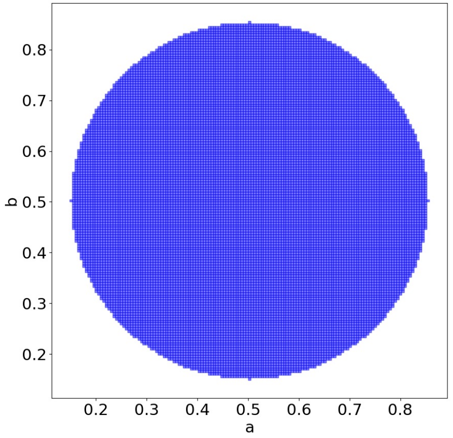


Applications

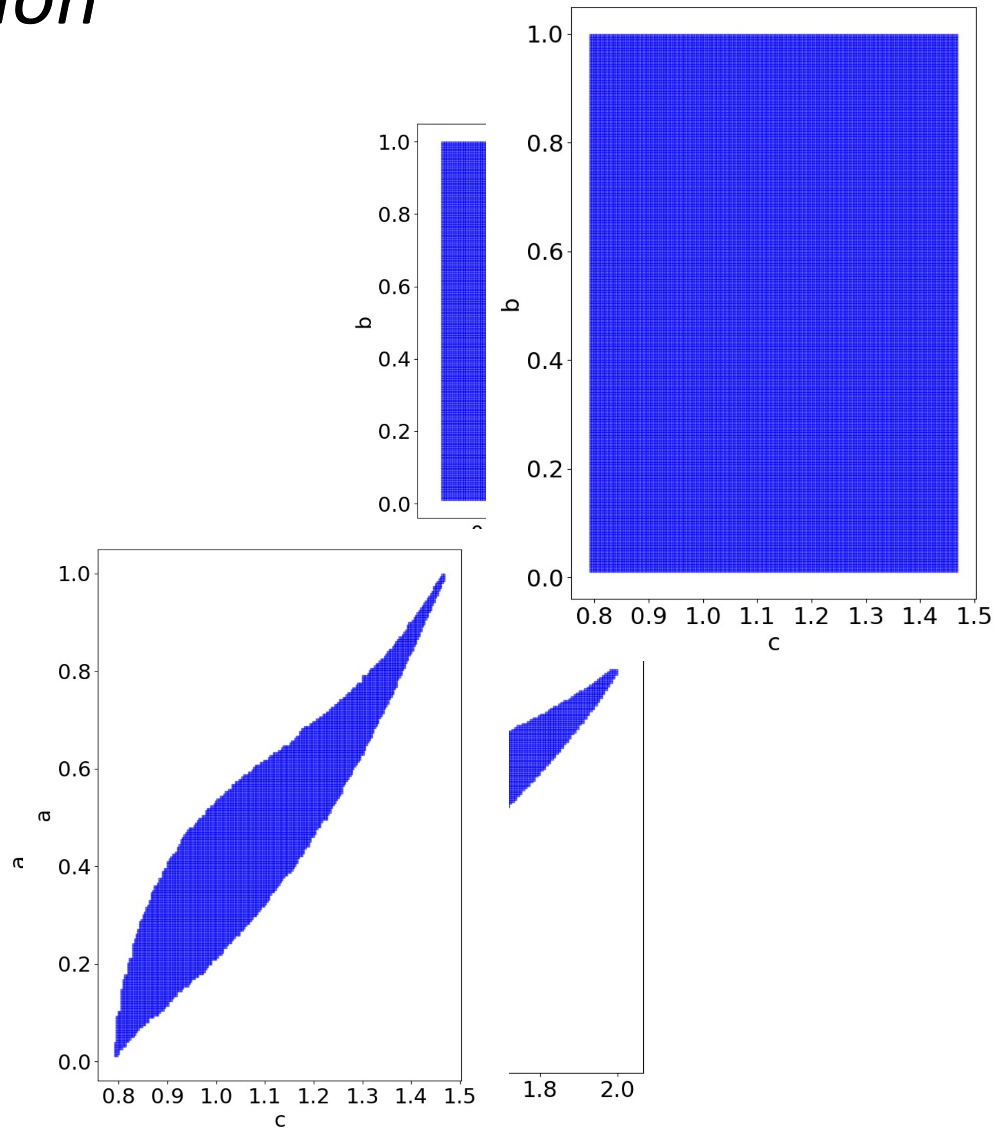
- Simple functions with repeated variables
- Interval projection in ODEs

Simple function

- $c = (a + b)^a$
- $a = [0.01, 1]$
- $b = [0.01, 1]$



- $c_{naive} = [0.2,2]$
- $c_{zone} = [0.67,2]$



With initial dependence

$\Rightarrow$ $c_{zone} = [0.79,1.47]$

Ordinary differential equations

- $\frac{dy}{dt} = f(t, y(t))$
- $y(0) = y_0$

Find $y(t)$

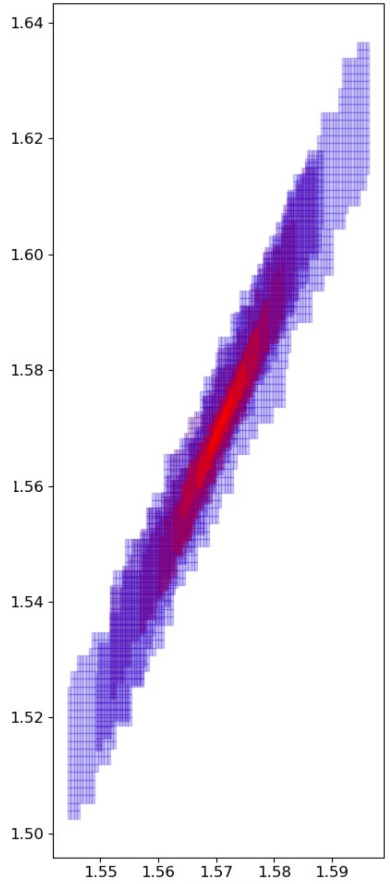
- Simplest way to integrate $f(t, y)$ is the Euler method
- Define time step *h*
- Approx: $y_{i+1} \approx \; y_i + h\, f(t_i, y_i)$

Repeated

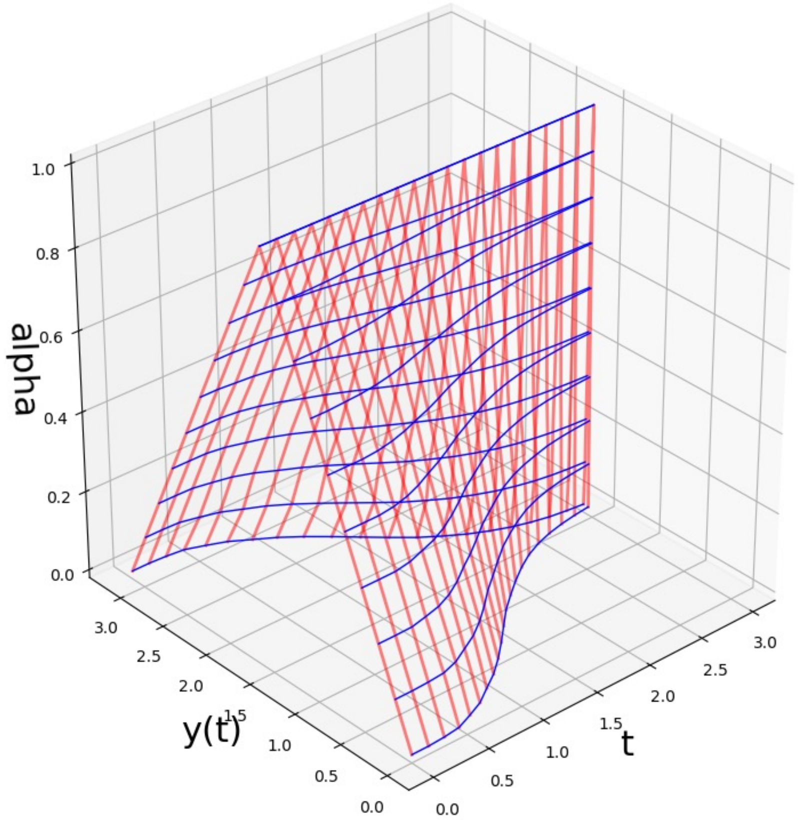
Ordinary differential equations

- $\frac{dy}{dt}(t, y) = t\cos(y)$
- $y_0 = [0,\ \pi]$
- $t \in [0, 3]$

$y(2.7s)$



$y(3s)$



Dominik Hose and Michael Hanss. "Possibilistic calculus as a conservative counterpart to probabilistic calculus." *Mechanical Systems and Signal Processing* 133 (2019): 106290.

Copulas and p-boxes

F_X, F_Y, C_{XY}

$$Z = X + Y$$

$$\tilde{Y} = Z - X$$

F_Z, F_X

C_{ZX}?

$$C_{ZX}(u,v) = \iint\limits_{D_{F_Z^{\wedge}(u), F_X^{\wedge}(v)}} dC_{XY}(F_X(x), F_Y(y))$$

$$D: \begin{cases} x + y < F_Z^{-1}(u) \\ x < F_X^{-1}(u) \end{cases}$$

Copulas and p-boxes

$F_X \, , F_Y , C_{XY}$

$$Z = X + Y$$
$$\tilde{Y} = Z - X$$

$F_Z \, , F_X$

$C_{ZX}?$

$$C_{ZX}(u+\delta, v+\delta) = C_{ZX}(u,v) - C_{ZX}(u+\delta, v) - C_{ZX}(u, v+\delta) + \iint_A dC_{XY}(F_X(x), F_Y(y))$$

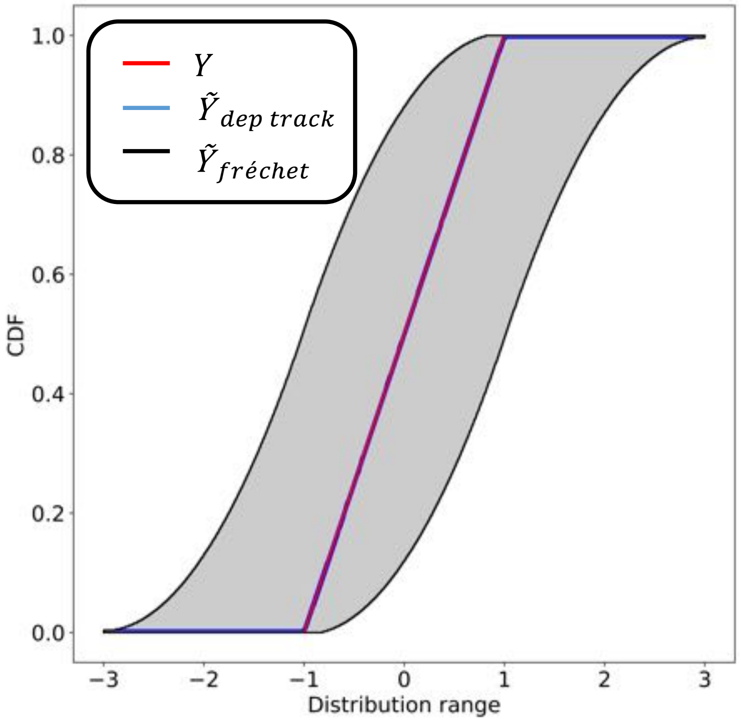
$$A : \begin{cases} F_Z{}^{-1}(u) \leq x + y < F_Z{}^{-1}(u+\delta) \\ F_X{}^{-1}(u) \leq x < F_X{}^{-1}(u+\delta) \end{cases}$$

Copulas and p-boxes

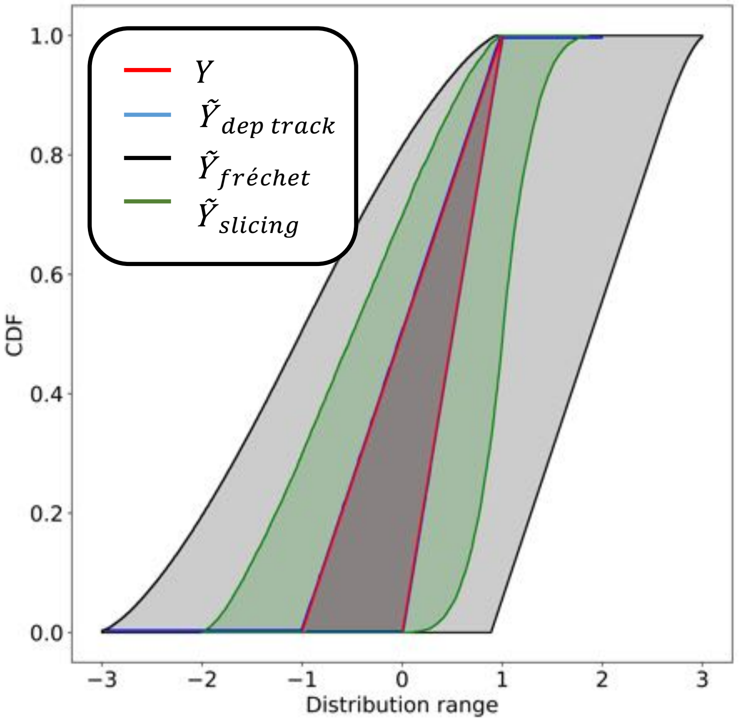
$$Z = X + Y$$

$$\tilde{Y} = Z - X$$

X = Y = U(-1, 1)



X = Y = U([-1,0], 1)



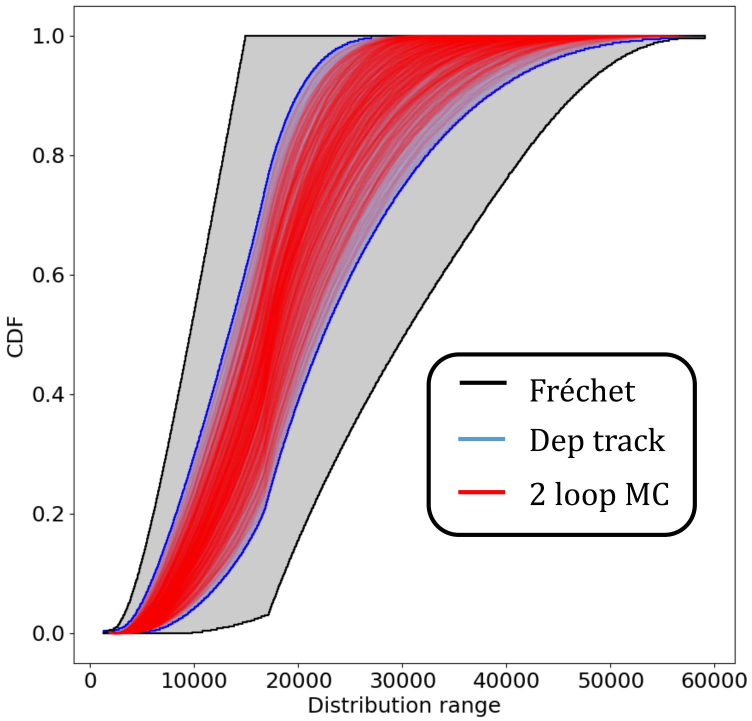
Copulas and p-boxes

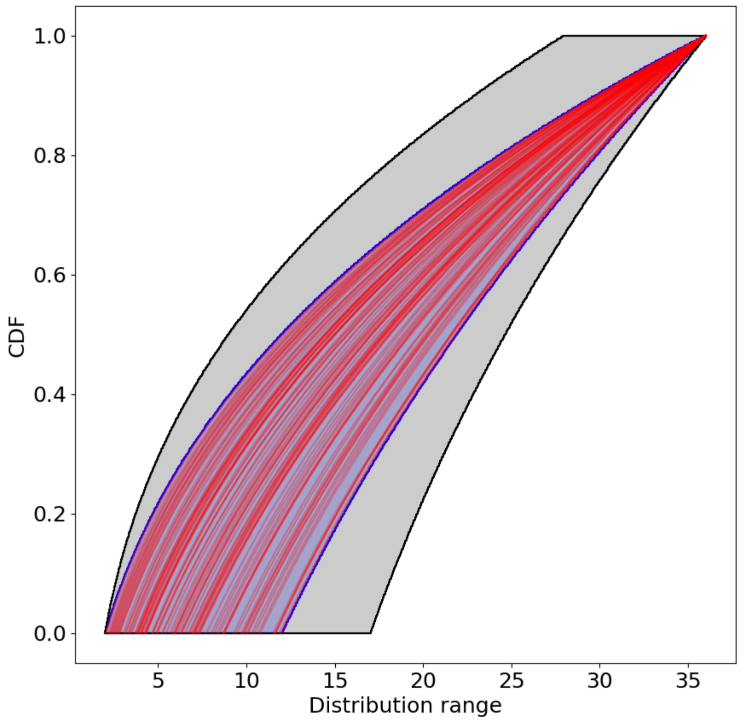
$$Z = (X+Y)^X,$$

$$X \sim Beta(3,[0.5,1]),\ Y \sim U(2,[3,4]),$$

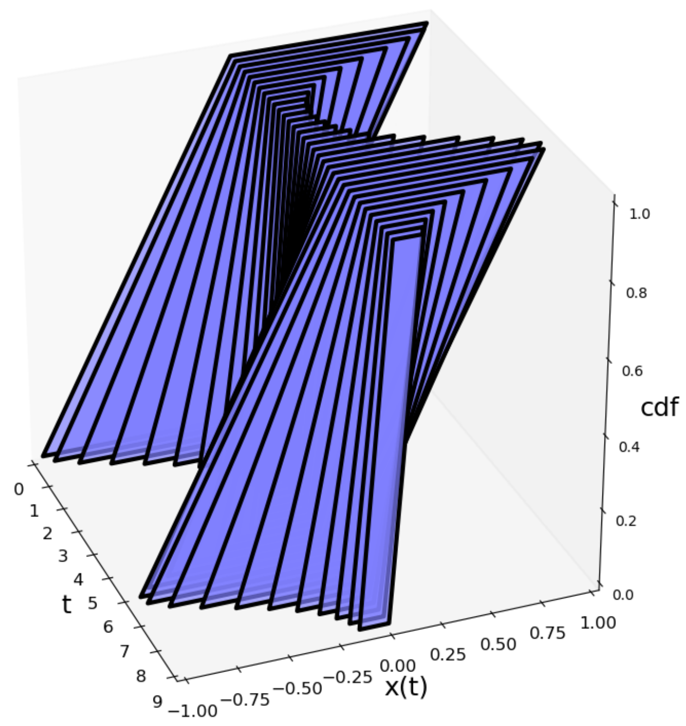
$$C_{XY} = Gau(cor = -0.5)$$

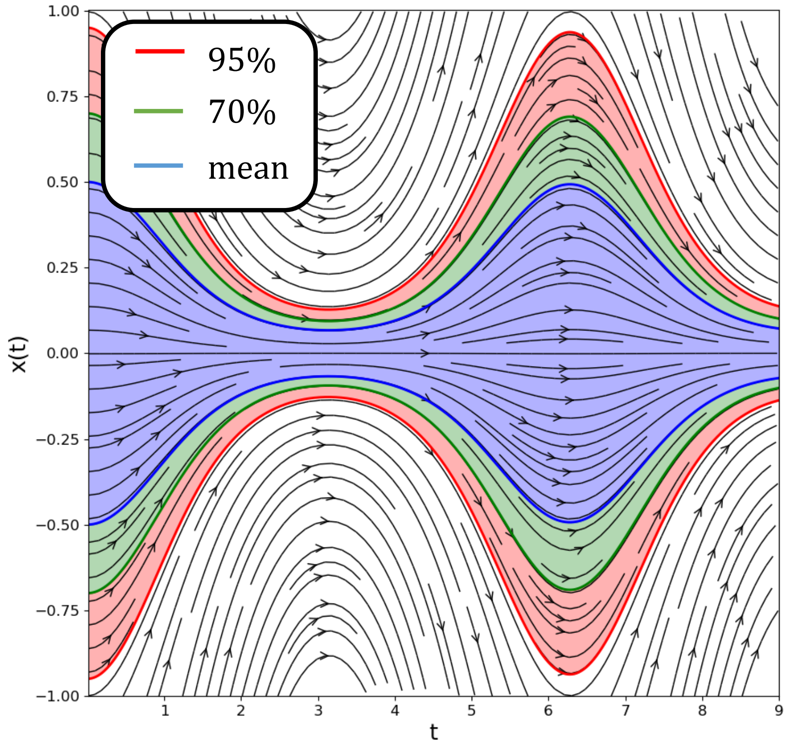
$$Z = X^3 + X^2, \text{ with } X \sim U([1,2],3)$$

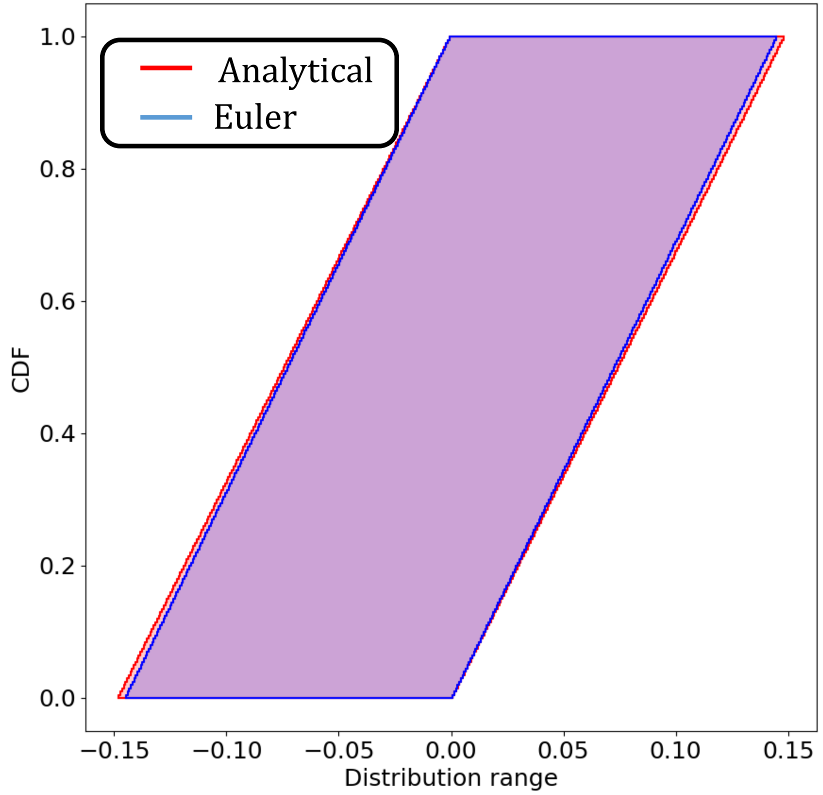




Copulas and p-boxes







Conclusions

- An extension to Interval Arithmetic to include/ track dependencies
- Dependencies stored in Boolean matrices
- We are only storing/working with bivariate dependencies, multivariate dependencies inferred from bivariate
- Same concept with copulas and p-boxes. But how do you project/inference?
- Julia packages:

  https://github.com/AnderGray/ZoneArithmetic.jl
  https://github.com/AnderGray/ProbabilityBoundsAnalysis.jl

go raibh míle maith agat