The interval (discrete) Fourier transform

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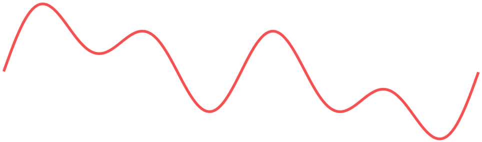
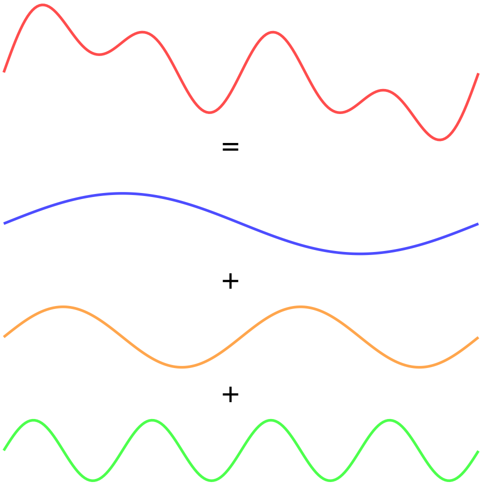
May 17, 2021

Outline

Motivations

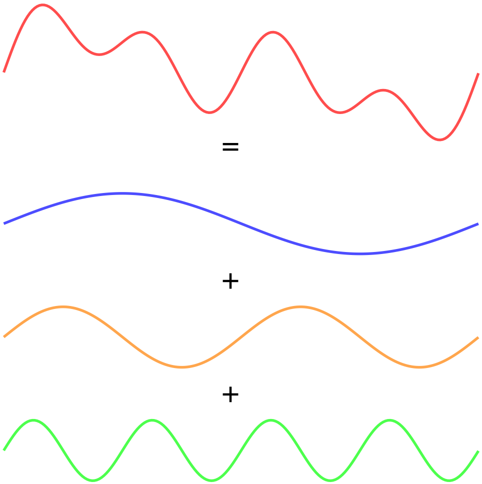


Motivations



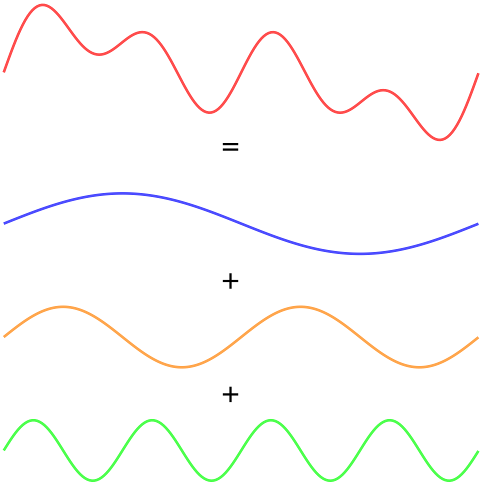
Motivations

- Imprecise/poor sensor measurements



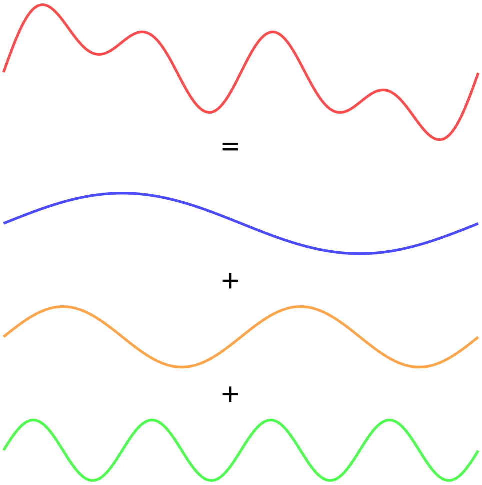
Motivations

- Imprecise/poor sensor measurements
- Missing data



Motivations

- Imprecise/poor sensor measurements
- Missing data
- Relaxed assumptions



Notation

- $\mathcal{F}$: Fourier transform
- $\mathcal{F}_h$: Fourier transform for a given harmonic h
- f: a function
- $[f]$: Natural extension
- $\hat{f}$: United extension

In this presentation we use capital letters to denote interval variables.

The discrete Fourier transform

$$\mathcal{F}_h : \mathbb{R}^n \to \mathbb{C}, \quad \forall h \in \mathbb{Z}_+$$

$$\mathcal{F}_h(x) := \sum_{k=0}^{n-1} x_k \; e^{-i\frac{2\pi}{n}hk}$$

The discrete Fourier transform

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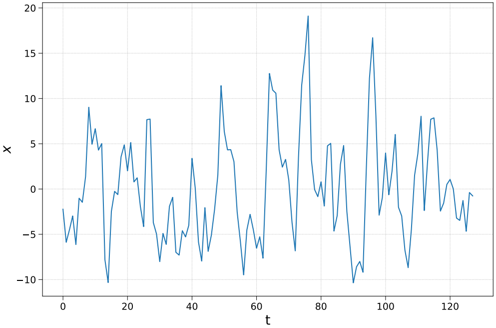
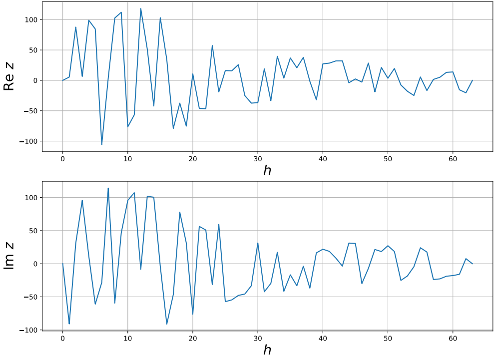
$$\mathcal{F}_h(x) := \sum_{k=0}^{n-1} x_k \; e^{-i \frac{2\pi}{n} hk}$$

or

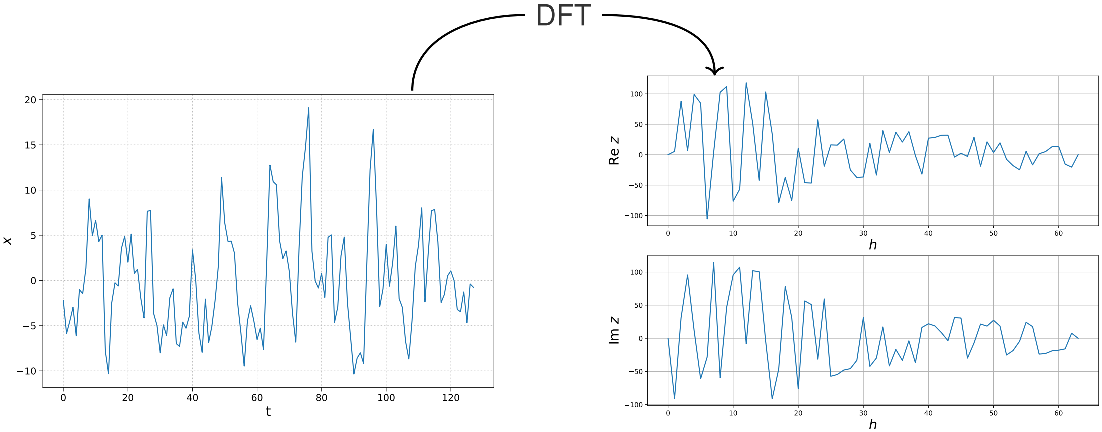
$$\mathcal{F}_h(x) := \sum_{k=0}^{n-1} x_k \; \left(\cos \frac{2\pi}{n} hk - i \sin \frac{2\pi}{n} hk \right)$$

The discrete Fourier transform

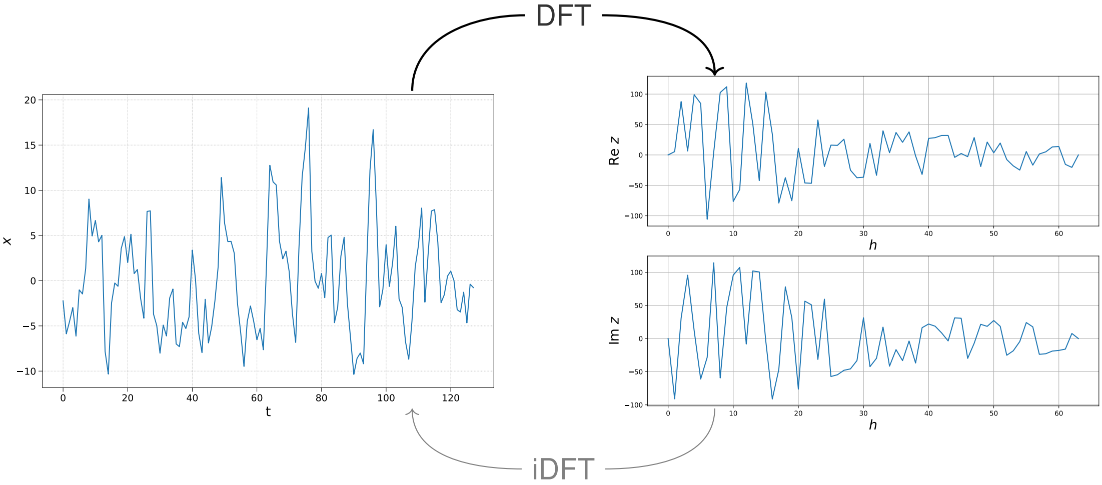




The discrete Fourier transform



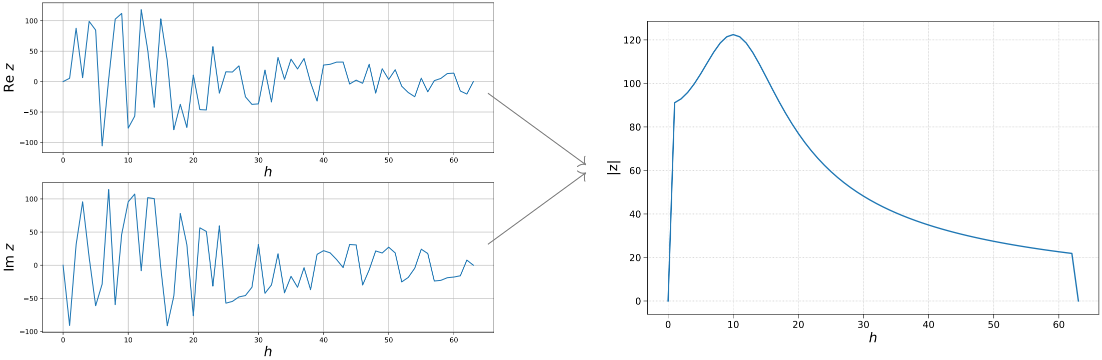
The discrete Fourier transform


DFT

iDFT

Amplitude of the Fourier transform

$$\left|\sum_{k=0}^{n-1} x_k\ e^{-{\color{red}i}\frac{2\pi}{n}hk}\right|$$



United extension $\hat{\mathcal{F}}$

$$\hat{\mathcal{F}} : S(A) \to S(B)$$

where, $S(A)$, $S(B)$ is the family of subset of A, B.

$$\hat{\mathcal{F}}(X) = \bigcup_{X \in S(A)} \{\mathcal{F}(X) \mid x \in X\}$$

United extensions are **inclusion monotonic**

Interval extension $[\mathcal{F}]$

$$\mathcal{F} : \mathbb{R}^n \to \mathbb{C}$$

$$[\mathcal{F}](\{x\}) = \mathcal{F}(x),$$

where, $\{x\}$ is a degenerate interval.

Fundamental theorems of IA

Theorem 1 (United extension)

If $[f]$ is an ***inclusion monotonic*** *extension of f then,*

$$\hat{\mathcal{F}}(x) \subseteq [\mathcal{F}](x). \tag{1}$$

Theorem 2 (Natural extension)

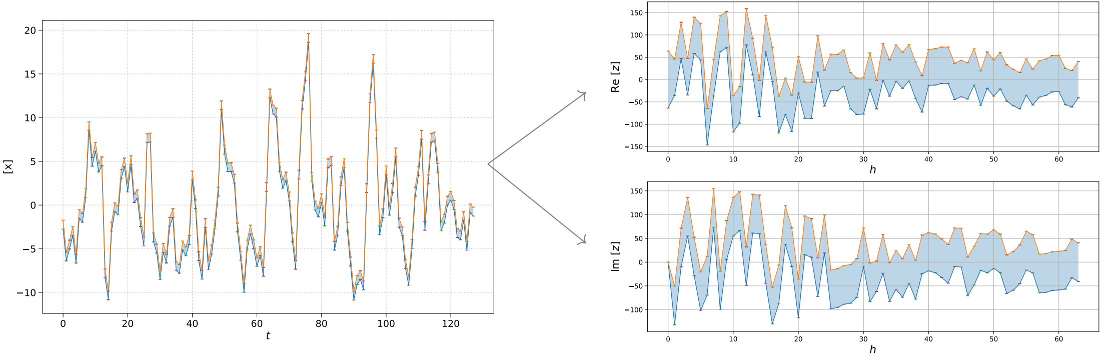
Any real function whose entries have been replaced with intervals is inclusion monotonic.

$$f(x) = x - x^2, \qquad [f](X) = X - X^2. \tag{2}$$

Interval discrete Fourier transform

▪

$$\sum_{k=0}^{n-1} X_k \, e^{-i\frac{2\pi}{n}hk}$$



Computing $[\mathcal{F}]$

$$[\mathcal{F}_h](x) = \sum_{k=0}^{n-1} X_k \; e^{-i\frac{2\pi}{n}hk}$$

$$X = [X_1, X_2] = \{x \in \mathbb{R}^n | X_1 \leq x \leq X_2\}$$

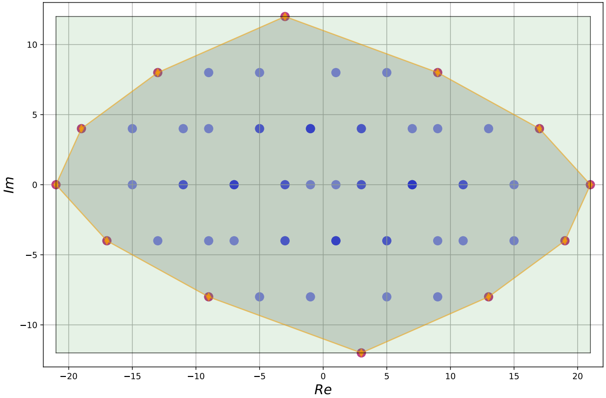
$$[\mathcal{F}] = \left[\inf_{X_1 \leq x \leq X_2} \mathcal{F}(x), \quad \sup_{X_1 \leq x \leq X_2} \mathcal{F}(x) \right]$$

For the united extension extra computational power is needed.

Example, $f : \mathbb{R}^6 \to \mathbb{C}, \quad f(x) = b\, x^T$

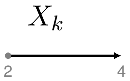
$$b = \begin{pmatrix} 1+i2 & -2+i2 & 3+i2 & \dots \\ 4-i2 & -5-i2 & 6+i2 & \end{pmatrix}$$

$$X^T = \begin{pmatrix} [-1,1] \\ [-1,1] \\ [-1,1] \\ [-1,1] \\ [-1,1] \\ [-1,1] \end{pmatrix}$$

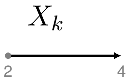


$$\sum_{k=0}^{n-1} X_k \left(\cos \frac{2\pi}{n} hk - i \sin \frac{2\pi}{n} hk \right)$$

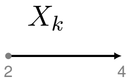
$$\sum_{k=0}^{n-1} X_k \left(\cos\frac{2\pi}{n}hk - i\sin\frac{2\pi}{n}hk\right)$$



$$\sum_{k=0}^{n-1} X_k \left(\underset{r_k}{\boxed{\cos \frac{2\pi}{n} hk}} - i \underset{i_k}{\boxed{\sin \frac{2\pi}{n} hk}} \right)$$



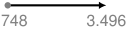
$$\sum_{k=0}^{n-1} X_k \left(\underbrace{\boxed{\cos \frac{2\pi}{n} hk}}_{r_k} - i \underbrace{\boxed{\sin \frac{2\pi}{n} hk}}_{i_k} \right)$$



$r_k = 0.874$

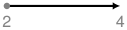
$i_k = 1.724$

$$\sum_{k=0}^{n-1} X_k \left(\underbrace{\boxed{\cos\frac{2\pi}{n}hk}}_{r_k} - {\color{red}i}\underbrace{\boxed{\sin\frac{2\pi}{n}hk}}_{i_k}\right)$$



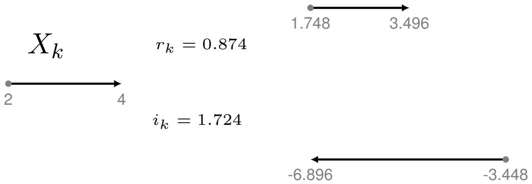
X_k

$r_k = 0.874$



$i_k = 1.724$

$$\sum_{k=0}^{n-1} X_k \left(\underset{r_k}{\boxed{\cos\frac{2\pi}{n}hk}} - i\,\underset{i_k}{\boxed{\sin\frac{2\pi}{n}hk}}\right)$$



X_k

$r_k = 0.874$

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$$\sum_{k=0}^{n-1} X_k \left(\underbrace{\boxed{\cos\frac{2\pi}{n}hk}}_{r_k} - {\color{red}i}\underbrace{\boxed{\sin\frac{2\pi}{n}hk}}_{i_k}\right)$$

X_k

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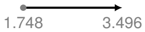
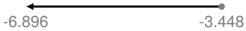
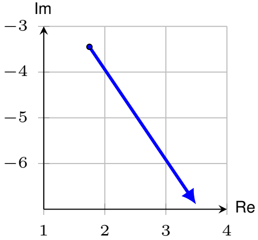


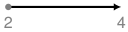
$i_k = 1.724$





$$\sum_{k=0}^{n-1} X_k \left(\underbrace{\boxed{\cos \frac{2\pi}{n} hk}}_{r_k} - i \underbrace{\boxed{\sin \frac{2\pi}{n} hk}}_{i_k} \right)$$

X_k



$r_k = 0.874$

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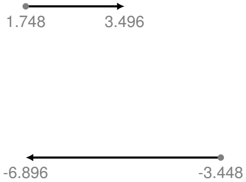
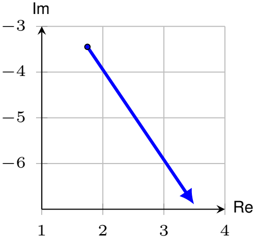




$$\sum_{k=0}^{n-1} X_k \cdot r_k + i\, X_k \cdot i_k, \quad r_k, i_k \in \mathbb{R}$$

$$\sum_{k=0}^{n-1} X_k \cdot r_k + i\, X_k \cdot i_k, \quad r_k, i_k \in \mathbb{R}$$

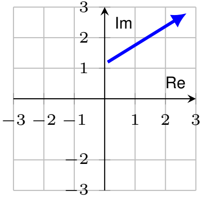
$$X_0 \cdot r_0 \;+\; X_1 \cdot r_1 \;+\; X_2 \cdot r_2 \;+\; \dots$$

$$\sum_{k=0}^{n-1} X_k \cdot r_k + i\, X_k \cdot i_k, \quad r_k, i_k \in \mathbb{R}$$

$$\begin{array}{lccccccc} & X_0 \cdot r_0 & + & X_1 \cdot r_1 & + & X_2 \cdot r_2 & + & \dots \\ i & X_0 \cdot i_0 & + & X_1 \cdot i_1 & + & X_2 \cdot i_2 & + & \dots \end{array}$$

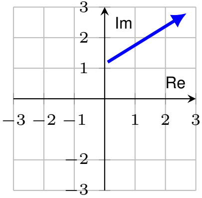
$$\sum_{k=0}^{n-1} X_k \cdot r_k + i\, X_k \cdot i_k, \quad r_k, i_k \in \mathbb{R}$$

$$\begin{array}{c|c|cccccc}
\cline{2-2}
 & X_0 \cdot r_0 & + & X_1 \cdot r_1 & + & X_2 \cdot r_2 & + & \ldots \\
i & X_0 \cdot i_0 & + & X_1 \cdot i_1 & + & X_2 \cdot i_2 & + & \ldots \\
\cline{2-2}
\end{array}$$



$$\sum_{k=0}^{n-1} X_k \cdot r_k + i\, X_k \cdot i_k, \quad r_k, i_k \in \mathbb{R}$$

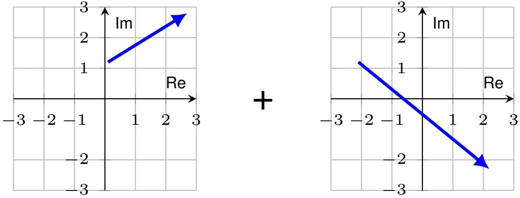
$$\begin{array}{ccccccccc} & \boxed{\begin{array}{c} X_0 \cdot r_0 \\ X_0 \cdot i_0 \end{array}} & + & X_1 \cdot r_1 & + & X_2 \cdot r_2 & + & \dots \\ i & & + & X_1 \cdot i_1 & + & X_2 \cdot i_2 & + & \dots \end{array}$$



+

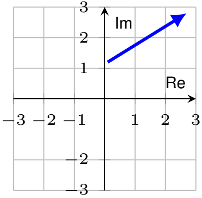
$$\sum_{k=0}^{n-1} X_k \cdot r_k + i\ X_k \cdot i_k, \qquad r_k, i_k \in \mathbb{R}$$

$$\begin{array}{c} \\ i \end{array} \boxed{\begin{array}{c} X_0 \cdot r_0 \\ X_0 \cdot i_0 \end{array}} \begin{array}{c} + \\ + \end{array} \boxed{\begin{array}{c} X_1 \cdot r_1 \\ X_1 \cdot i_1 \end{array}} \begin{array}{c} + \\ + \end{array} \begin{array}{c} X_2 \cdot r_2 \\ X_2 \cdot i_2 \end{array} \begin{array}{c} + \\ + \end{array} \begin{array}{c} \dots \\ \dots \end{array}$$



$$\sum_{k=0}^{n-1} X_k \cdot r_k + i\ X_k \cdot i_k, \quad r_k, i_k \in \mathbb{R}$$

$$\begin{matrix} \\ i \end{matrix} \boxed{\begin{matrix} X_0 \cdot r_0 \\ X_0 \cdot i_0 \end{matrix}} \begin{matrix} + \\ + \end{matrix} \boxed{\begin{matrix} X_1 \cdot r_1 \\ X_1 \cdot i_1 \end{matrix}} \begin{matrix} + \\ + \end{matrix} \begin{matrix} X_2 \cdot r_2 \\ X_2 \cdot i_2 \end{matrix} \begin{matrix} + \\ + \end{matrix} \begin{matrix} \dots \\ \dots \end{matrix}$$



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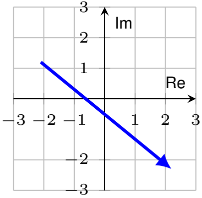
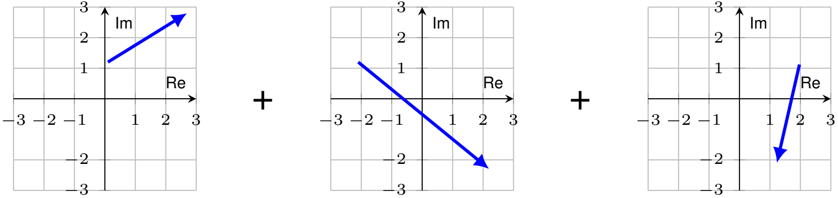


+

$$\sum_{k=0}^{n-1} X_k \cdot r_k + {\color{red}i}\ X_k \cdot i_k, \quad r_k, i_k \in \mathbb{R}$$

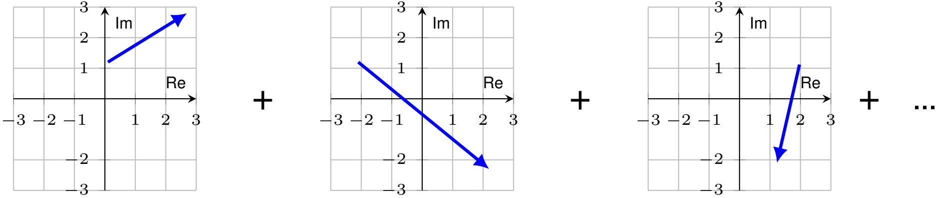
$$\begin{array}{c} \\ {\color{red}i} \end{array}\ \boxed{\begin{array}{c} X_0 \cdot r_0 \\ X_0 \cdot i_0 \end{array}} \begin{array}{c} + \\ + \end{array} \boxed{\begin{array}{c} X_1 \cdot r_1 \\ X_1 \cdot i_1 \end{array}} \begin{array}{c} + \\ + \end{array} \boxed{\begin{array}{c} X_2 \cdot r_2 \\ X_2 \cdot i_2 \end{array}} \begin{array}{c} + \\ + \end{array} \begin{array}{c} \dots \\ \dots \end{array}$$



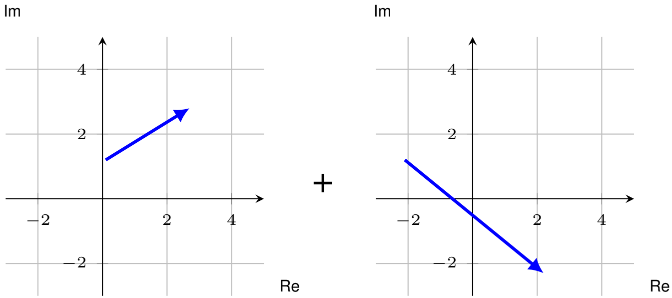
$$\sum_{k=0}^{n-1} X_k \cdot r_k + i \; X_k \cdot i_k, \qquad r_k, i_k \in \mathbb{R}$$

$$\begin{array}{cccccccc} & \boxed{\begin{array}{c} X_0 \cdot r_0 \\ X_0 \cdot i_0 \end{array}} & \begin{array}{c} + \\ + \end{array} & \boxed{\begin{array}{c} X_1 \cdot r_1 \\ X_1 \cdot i_1 \end{array}} & \begin{array}{c} + \\ + \end{array} & \boxed{\begin{array}{c} X_2 \cdot r_2 \\ X_2 \cdot i_2 \end{array}} & \begin{array}{c} + \\ + \end{array} & \begin{array}{c} \dots \\ \dots \end{array} \\ \end{array}$$

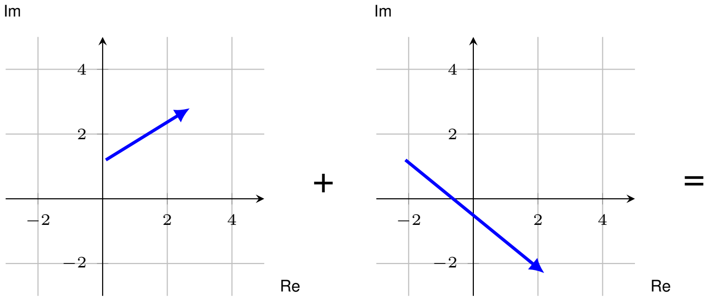
with the lower row prefixed by i.



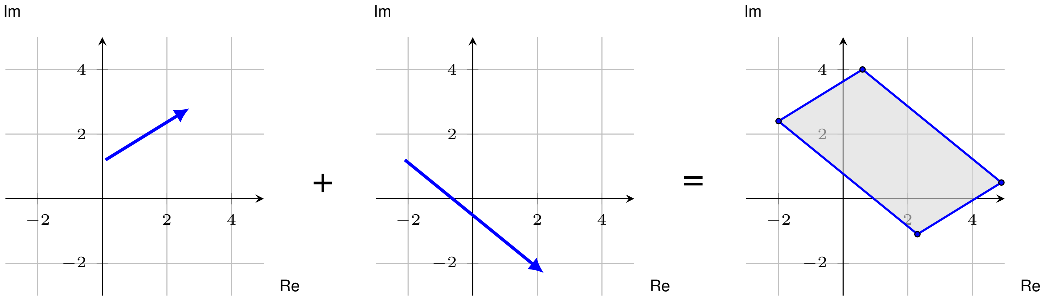
Sum of dependent intervals



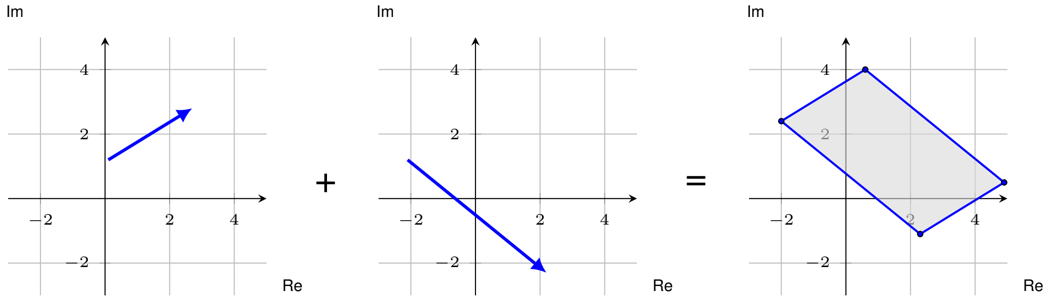
Sum of dependent intervals



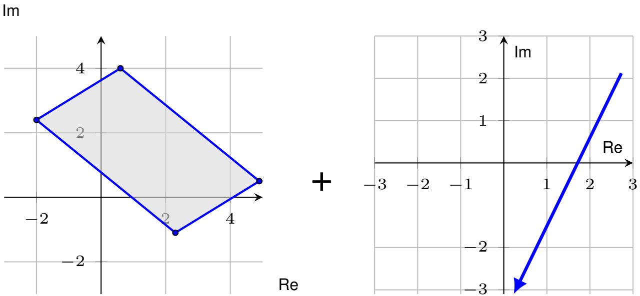
Sum of dependent intervals



Sum of dependent intervals



The sum is not closed w.r.t. the diagonals.


Im
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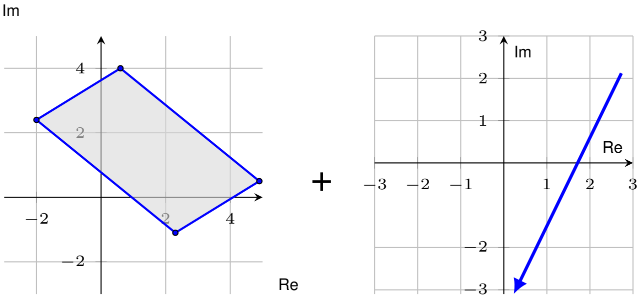
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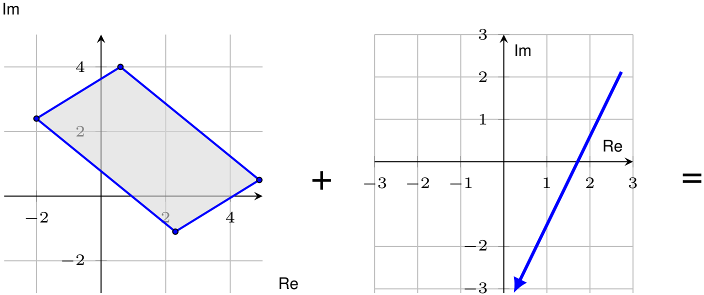
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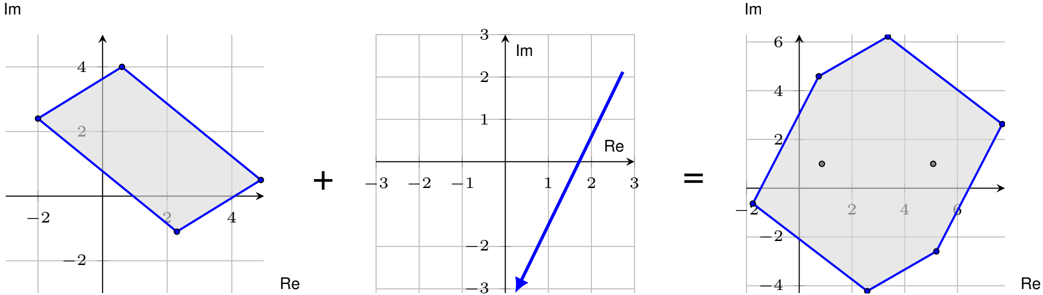
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Main theorems (Minkowski addition)

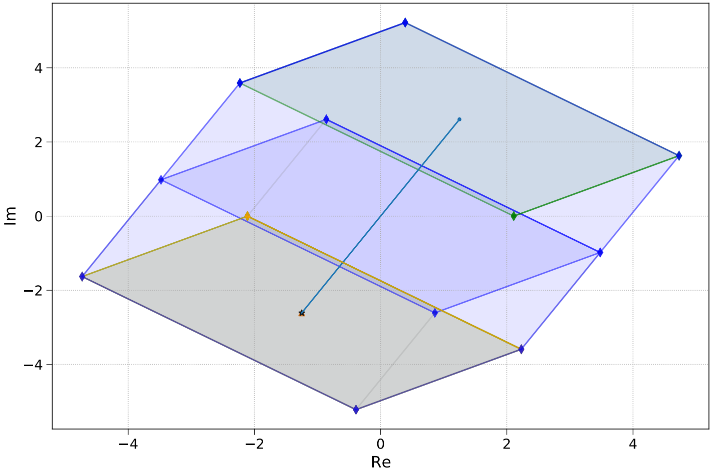
Theorem 3 (Linear transformation of convex sets)

Let $A \subseteq \mathbb{R}^n$ *be a convex set and suppose that* f *is the linear map of* $f : \mathbb{R}^n \to \mathbb{R}^m$*. Then* f *maps the extreme points of* A *onto the extreme points of* $f(A)$*.*

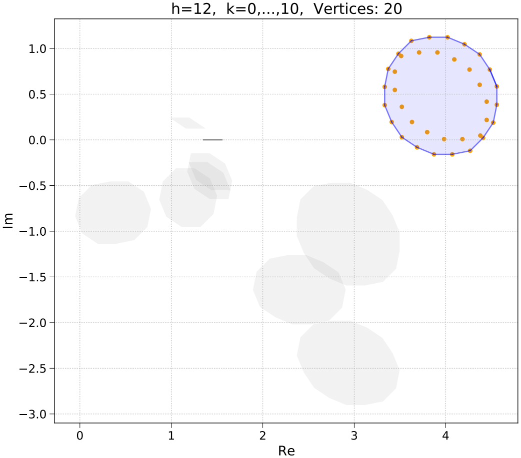
Theorem 4 (Sum of a convex set plus a segment)

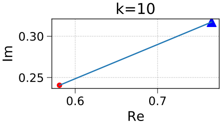
The maximum distance between a convex polygon and a segment is attained at the endpoints of the segment.

Proof (intuition)

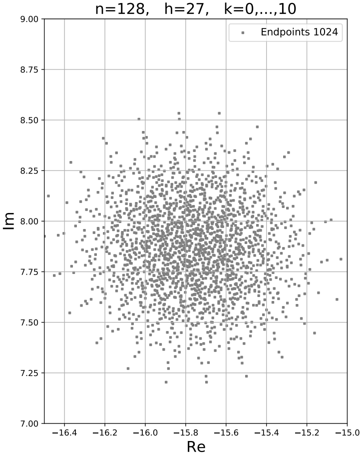


Algorithm

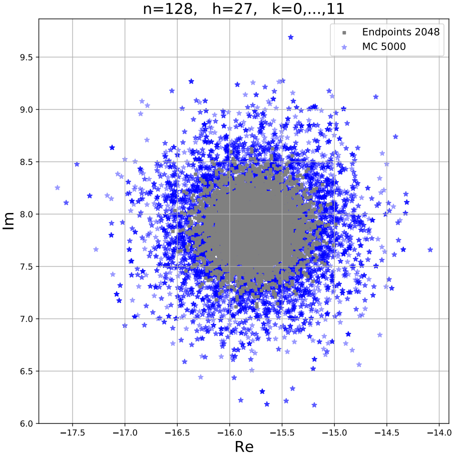

h=12, k=0,...,10, Vertices: 20
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
Im
0
1
2
3
4
Re


k=10
0.30
0.25
Im
0.6
0.7
Re

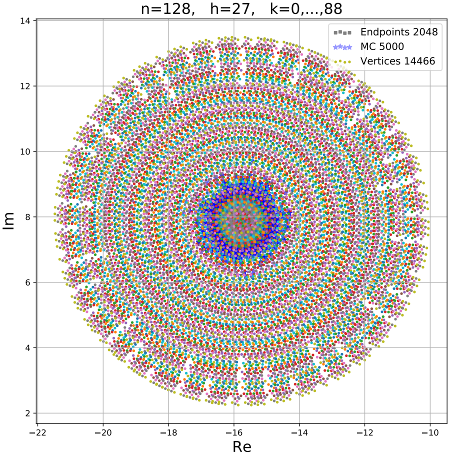
Centered form and MC samples



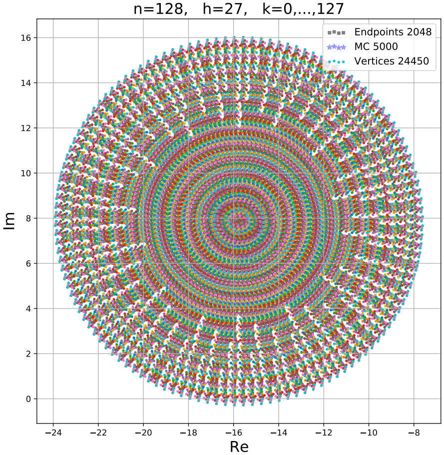
Centered form and MC samples



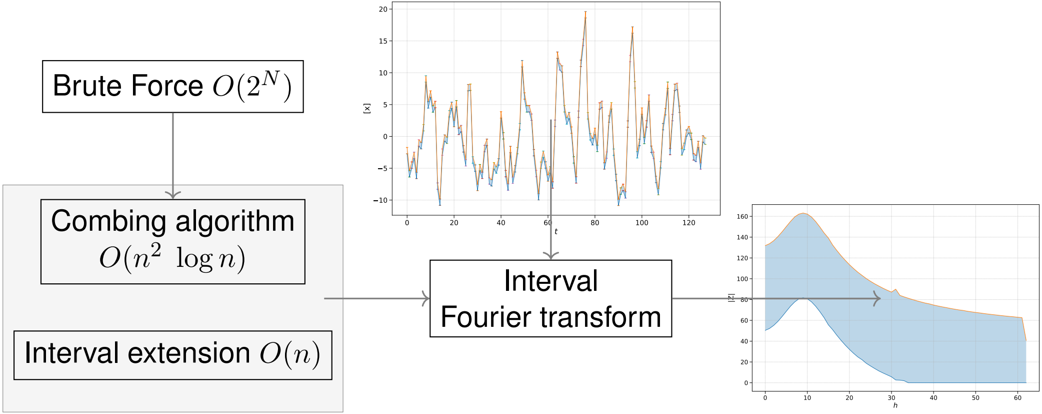
Centered form and MC samples



Centered form and MC samples



Computational cost and summary



Brute Force $O(2^N)$

Combing algorithm $O(n^2\ \log n)$

Interval extension $O(n)$

Interval Fourier transform