9TH INTERNATIONAL WORKSHOP ON RELIABLE ENGINEERING COMPUTING
Risk and Uncertainty in Engineering Computations
Virtual Conference: May 17-20, 2021
REC2021

Università degli Studi di Palermo

STOCHASTIC RESPONSE OF BEAMS EQUIPPED WITH TUNED MASS DAMPERS, CONSIDERING SPRING INERTIAL EFFECTS, SUBJECTED TO POISSONIAN LOADS

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The research presented in this presentation was carried out as part of the H2020-MSCA-ETN-2016. This project has received funding from the European Union's H2020 Programme for research, technological development and demonstration under grant agreement number 721493

MOTIVATIONS AND OUTLINE

Motivations

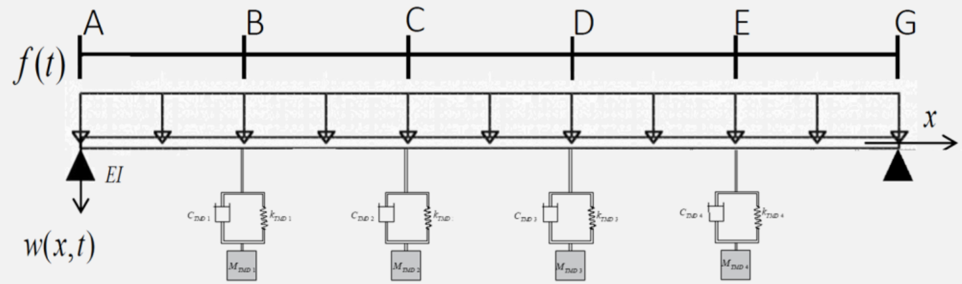
- Robust mathematical modelling is of ever-increasing importance as a means to evaluate the dynamic response of bridges under design and to perform monitoring of existing bridges. For this, the Euler-Bernoulli beam model is typically used.
- The ability to perform "virtual experiments", in which an accurate, computationally efficient mathematical model can predict the response of a bridge to loading, is also of increasing importance as a means to evaluate different configurations easily and quickly during the design phase. This requires highly accurate mathematical models to be applicable.
- Finally, as more slender, sustainable structures are being designed, a robust, exact analytical method is required to analyse their feasibility and classical methods are computationally intensive, the proposed method addresses this limitation.

Outline

- The Classical Method
- The Proposed Method – with the helical spring model
- Poissonian Loading
- Numerical Application and Discussion
- Conclusions

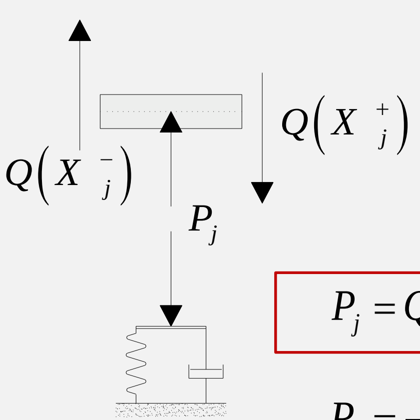
CLASSICAL SOLUTION OF THE MULTI-SPAN EULER-BERNOULLI BEAM

Each span is treated individually and is related to adjoining spans by modifying the boundary conditions



This, therefore, results in large, complex matrices which require large amounts of computational time to solve

The figure above shows a beam of just 5 spans, using the classical method this would require 5 equations of motion, generating 20 unknown constants all of which are coupled and must be solved simultaneously



$$\boxed{P_j = Q\left(X_j^-\right) - Q\left(X_j^+\right)}$$

$$P_j = -K_{\text{TMD J}}\, w\left(X_j\right)$$

$$EI\frac{\partial^4 w_{AB}(x,t)}{\partial x^4} + \overline{m}\frac{\partial^2 w_{AB}(x,t)}{\partial t^2} = f(t)$$

$$\vdots$$

$$EI\frac{\partial^4 w_{EG}(x,t)}{\partial x^4} + \overline{m}\frac{\partial^2 w_{EG}(x,t)}{\partial t^2} = f(t)$$

CLASSICAL SOLUTION OF THE MULTI-SPAN EULER BERNOULLI BEAM – FREE VIBRATION

The characteristic equation then becomes:

$$EI\frac{\partial^4 w_{AB}(x,t)}{\partial x^4}+\overline{m}\frac{\partial^2 w_{AB}(x,t)}{\partial t^2}=0$$

$$\vdots$$

$$EI\frac{\partial^4 w_{EG}(x,t)}{\partial x^4}+\overline{m}\frac{\partial^2 w_{EG}(x,t)}{\partial t^2}=0$$

Applying the separate variables approach:

$$w(x,t)=\psi(x)e^{i\omega t}$$

$$\theta(x,t)=\vartheta(x)e^{i\omega t}$$

$$M(x,t)=\mu(x)e^{i\omega t}$$

$$Q(x,t)=\chi(x)e^{i\omega t}$$

Where the eigenfunctions are related in the following manner:

$$\vartheta(x)=\frac{d\psi(x)}{dx}$$

$$\mu(x)=-\frac{d\vartheta(x)}{dx}$$

$$\chi(x)=\frac{d\mu(x)}{dx}$$

$$\frac{d\chi(x)}{dx}+\sigma^2\psi(x)=0$$

$$\frac{d^4\psi(x)}{dx^4}-\sigma^2\psi(x)=0$$

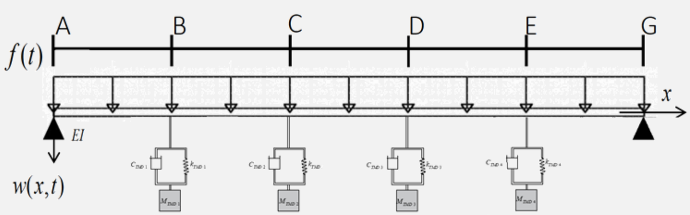
The characteristic equation (in space domain) can be expressed as:

$$\frac{d^4\psi_{AB}(x)}{dx^4}-\sigma^2\psi_{AB}(x)=0$$

$$\vdots$$

$$\frac{d^4\psi_{EG}(x)}{dx^4}-\sigma^2\psi_{EG}(x)=0$$

Where: $\sigma^2=\omega^2\overline{m}L^4/EI$



CLASSICAL SOLUTION TO THE MULTI-SPAN EULER BERNOULLI BEAM – FREE VIBRATION

Y (x) is a vector of the eigenfunctions of each response variable

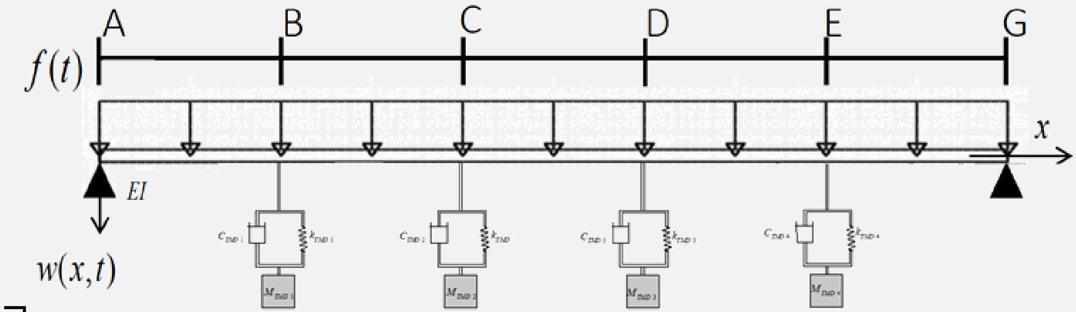
$$\mathbf{Y}(x)=\begin{bmatrix}\psi(x) & \vartheta(x) & \mu(x) & \chi(x)\end{bmatrix}^T$$

This can also be cast in the following form:

$$\mathbf{Y}(x)=\mathbf{\Omega}(x)\mathbf{c}$$

Where:

$$\mathbf{\Omega}(x)=\begin{bmatrix}\Omega_{\psi_1} & \Omega_{\psi_2} & \Omega_{\psi_3} & \Omega_{\psi_4}\\ \Omega_{\vartheta_1} & \Omega_{\vartheta_2} & \Omega_{\vartheta_3} & \Omega_{\vartheta_4}\\ \Omega_{\mu_1} & \Omega_{\mu_2} & \Omega_{\mu_3} & \Omega_{\mu_4}\\ \Omega_{\chi_1} & \Omega_{\chi_2} & \Omega_{\chi_3} & \Omega_{\chi_4}\end{bmatrix}$$

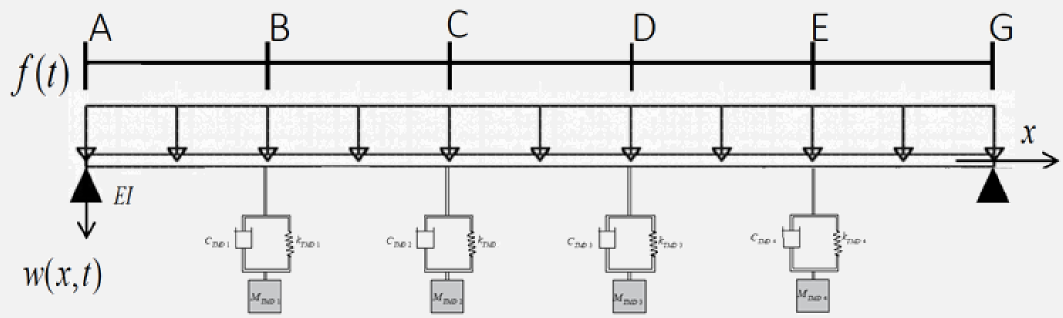


And where:

$$\mathbf{c}=\begin{bmatrix}c_1 & c_2 & c_3 & c_4\end{bmatrix}^T$$

CLASSICAL SOLUTION TO THE MULTI-SPAN EULER BERNOULLI BEAM - CONTINUED

$$\mathbf{\Omega}(x)=\begin{bmatrix}\Omega_{\psi_1} & \Omega_{\psi_2} & \Omega_{\psi_3} & \Omega_{\psi_4}\\ \Omega_{\vartheta_1} & \Omega_{\vartheta_2} & \Omega_{\vartheta_3} & \Omega_{\vartheta_4}\\ \Omega_{\mu_1} & \Omega_{\mu_2} & \Omega_{\mu_3} & \Omega_{\mu_4}\\ \Omega_{\chi_1} & \Omega_{\chi_2} & \Omega_{\chi_3} & \Omega_{\chi_4}\end{bmatrix}$$



$$\Omega_{\psi_1}(x)=e^{-\sigma x} \qquad \Omega_{\psi_2}(x)=e^{\sigma x} \qquad \Omega_{\psi_3}(x)=\cos(\sigma x) \qquad \Omega_{\psi_4}(x)=\sin(\sigma x)$$

$$\Omega_{\vartheta_1}(x)=-\sigma e^{-\sigma x} \qquad \Omega_{\vartheta_2}(x)=\sigma e^{\sigma x} \qquad \Omega_{\vartheta_3}(x)=-\sigma\sin(\sigma x) \qquad \Omega_{\vartheta_4}(x)=\sigma\cos(\sigma x)$$

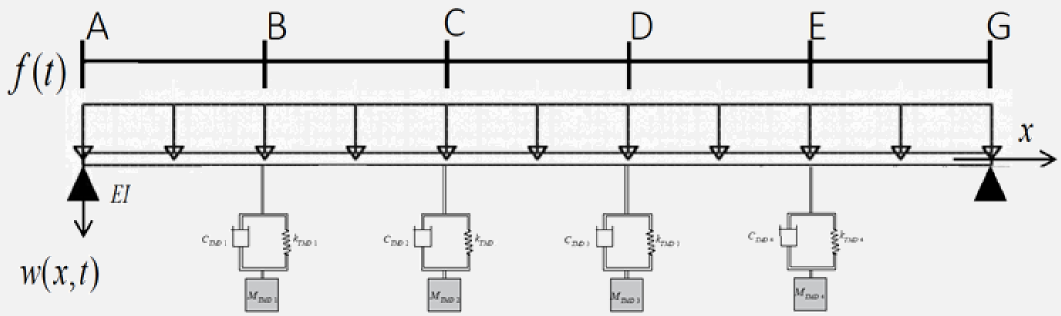
$$\Omega_{\mu_1}(x)=-\sigma^2 e^{-\sigma x} \qquad \Omega_{\mu_2}(x)=\sigma^2 e^{\sigma x} \qquad \Omega_{\mu_3}(x)=\sigma^2\cos(\sigma x) \qquad \Omega_{\mu_4}(x)=\sigma^2\sin(\sigma x)$$

$$\Omega_{\chi_1}(x)=\sigma^3 e^{-\sigma x} \qquad \Omega_{\chi_2}(x)=-\sigma^3 e^{\sigma x} \qquad \Omega_{\chi_3}(x)=-\sigma^3\sin(\sigma x) \qquad \Omega_{\chi_4}(x)=\sigma^3\cos(\sigma x)$$

Where n number of matrices are required for j discontinuities

$$\boxed{n=j+1}$$

CLASSICAL SOLUTION TO THE MULTI-SPAN EULER BERNOULLI BEAM – RESPONSE TO A FORCING ACTION



Considering a distributed load for k modes:

$$w(x,\tau)=\sum_{k=1}^{\infty}\psi_k(x)\frac{\int_0^L \psi_k(x)\,dx}{M_k\quad\omega_k}\int_0^t f(\tau)\ \mathrm{Sin}(\omega_k(\tau-\mathrm{t}))\,d\tau$$

This response is built from Duhamel's integral where, at the kth mode:

ω_k Is the natural frequency

M_k Is the modal mass

$f(\tau)$ Is the forcing action

Where: $$M_k=\overline{m}\int_0^L \psi_k(x)^2\ dx$$

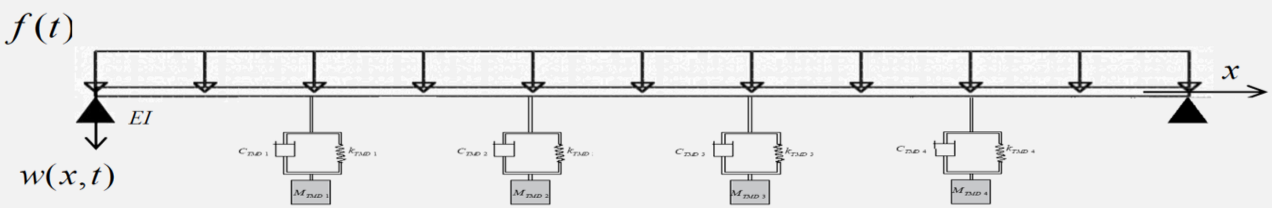
This method is very computationally taxing

PROPOSED METHOD – FREE VIBRATION

- The method put forward models each discontinuity as a reactionary force acting on the beam at a specific point.
- It is based on the theory of distributions to derive:
  - Closed form expressions of the eigenfunctions
  - The characteristic equation as the determinant of a 4x4 matrix, for any number of discontinuities
  - The orthogonality conditions for the eigenfunctions
  - The time-domain solution under any forcing action
- The Equation of Motion becomes:

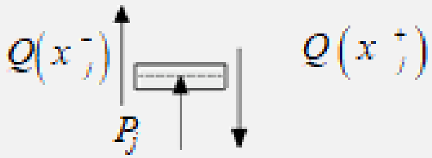
$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\bar{m}\frac{\partial^2 w(x,t)}{\partial t^2}+R(x,t)=0$$

Where: $$R(x,t)=\sum_{j=1}^{N}-P_j(t)\;\delta(x-x_j)$$



PROPOSED METHOD – FREE VIBRATION

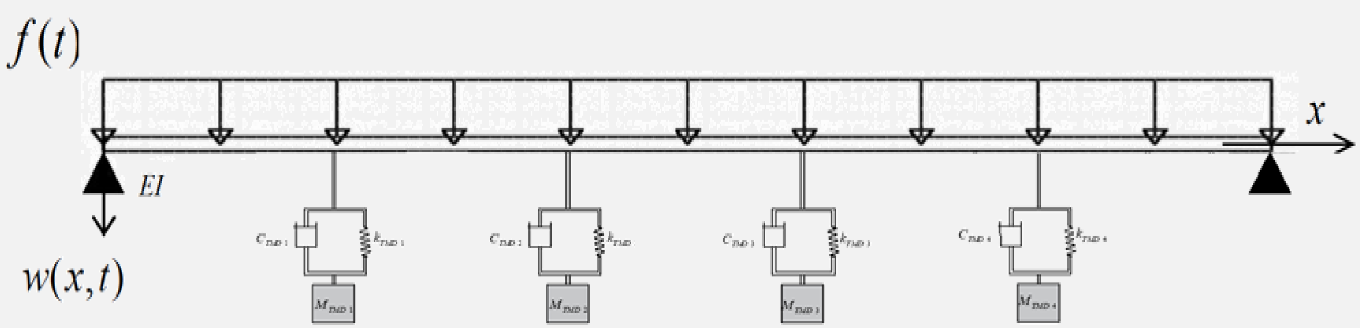
The characteristic equation considering the first eigenfunction becomes:



$$\frac{d^4\psi(x)}{dx^4}+\sum_{j=1}^{N}P_j(x,\omega)-\sigma^2\psi(x)=0$$

Where:

$$P_j(x,\omega)=-\varphi_j(\omega)\delta(x-x_j) \quad \text{and} \quad \varphi_j(\omega)=-K_{TMD\,j}(\omega)\ \psi(x_j)$$



PROPOSED METHOD – DEFINITION OF THE REACTIONARY FORCE

Previously the reactionary force has been defined as:

$$P_j(x,\omega) = -\varphi_j(\omega)\,\delta(x - x_j) \quad \text{and} \quad \varphi_j(\omega) = -K_{TMD\,j}(\omega)\;\psi(x_j)$$

Where:

$$K_{TMD\,j}(\omega) = \sqrt{EA_{\text{spring j}}}\sqrt{\rho_{\text{spring j}}}\;\omega\left(\frac{\omega\,\mathrm{M}_{TMD\text{ j}}\,\mathrm{Cos}\left(\frac{h_j\omega\sqrt{\rho_{\text{spring j}}}}{\sqrt{EA_{\text{spring j}}}}\right)+\sqrt{EA_{\text{spring j}}}\sqrt{\rho_{\text{spring j}}}\,\mathrm{Sin}\left(\frac{h_j\omega\sqrt{\rho_{\text{spring j}}}}{\sqrt{EA_{\text{spring j}}}}\right)}{-\sqrt{EA_{\text{spring j}}}\sqrt{\rho_{\text{spring j}}}\,\mathrm{Cos}\left(\frac{h_j\omega\sqrt{\rho_{\text{spring j}}}}{\sqrt{EA_{\text{spring j}}}}\right)+\omega\,\mathrm{M}_{TMD\text{ j}}\,\mathrm{Sin}\left(\frac{h_j\omega\sqrt{\rho_{\text{spring j}}}}{\sqrt{EA_{\text{spring j}}}}\right)}\right)$$

Where:

$\mathrm{M}_{TMD\,j}$ is the magnitude of the lumped mass of the jth TMD,

$EA_{\text{spring j}}$ the spring stiffness of the jth TMD's spring,

$\rho_{\text{spring j}}$ is the distributed mass of the spring which forms part of the jth TMD,

h_j is the spring length, and

ω is the frequency.

This definition of the reactionary force applies in cases where the TMD consists of just a spring mass and the spring element is not assumed to be massless.

$$\lim_{\rho_{\text{spring j}}\longrightarrow 0} K_{TMD\,j}(\omega) = \left(\frac{\mathrm{M}_{TMD\text{ j}}\;\omega^2\left(\frac{EA_{\text{spring j}}}{h_j}\right)}{\omega^2\;\mathrm{M}_{TMD\text{ j}} - \left(\frac{EA_{\text{spring j}}}{h_j}\right)}\right)$$

To the right is the limit case showing the reactionary force of a traditional mass-spring attachment

PROPOSED METHOD – FREE VIBRATION

The matrix method solution is also modified to account for the discontinuities:

$$\mathbf{Y}(x)=\mathbf{\Omega}(x)\mathbf{c}+\boxed{\sum_{j=1}^{N}\mathbf{J}(x,x_j)\,P_j(\omega)}$$

$$\mathbf{J}(x,x_j)=\begin{bmatrix} J_{\psi}^{(p)} \\ J_{\vartheta}^{(p)} \\ J_{\mu}^{(p)} \\ J_{\chi}^{(p)} \end{bmatrix}$$

Where the homogemeous solution is of the form:

$$\mathbf{\Omega}_{\psi}(x)=\begin{bmatrix}\Omega_{\psi_1} & \Omega_{\psi_2} & \Omega_{\psi_3} & \Omega_{\psi_4}\end{bmatrix} \qquad \mathbf{c}=\begin{bmatrix}c_1 & c_2 & c_3 & c_4\end{bmatrix}^T$$

Giving the particular solution:

$$J_{\psi}^{(p)}(x)=1/2\ \omega^{-3/2}\left[\sinh\left(\sqrt{\omega}\,(x-x_j)\right)-\sin\left(\sqrt{\omega}\,(x-x_j)\right)\right]H(x-x_j)$$

Where $\psi_j(\omega)$ is an unknown

Enforcing the boundary conditions leads to the characteristic equation as the determinant of a 4x4 matrix. Where $\mathbf{B}$ is a matrix constructed from enforcing B.Cs on the matrix $\mathbf{Y}(x)$

$$\mathbf{B}_{4\times4}\mathbf{c}_{4\times1}=\mathbf{0}_{4\times1} \qquad \det(\mathbf{B}_{4\times4})=\mathbf{0}$$

Adam, C., Di Lorenzo, S., Failla, G. & Pirrotta, G., 2017. On the Moving Load Problem in Beam Structures Equipped with Tuned Mass Dampers. *Meccanica,* Volume 52, pp. 3101-3115.

Sing, M.P., Moreschi, L.M., 2002. Optimal placement of dampers for passive response control. Earth Eng Struct Dyn 31:955–976

MULTI-SPAN RESPONSE TO A FORCING ACTION – ORTHOGONALITY CONDITIONS

Firstly, the equation of motion concerning free vibration is considered:

$$\frac{d^4\psi_k(x)}{dx^4} - \sigma_k{}^2\psi_k(x) - \sum_{j=1}^{N} K_{TMD\ J}(\omega_k)\psi_k(x_j) = 0$$

The orthogonality conditions are found by mulitplying the equation of motion at mode n by $\psi_m(x)$ and the equation of motion at mode m by $\psi_n(x)$:

$$\int_0^L \frac{d^2\psi_n(x)}{dx^2}\frac{d^2\psi_m(x)}{dx^2}dx - \sigma_n{}^2\int_0^L \psi_n(x)\psi_m(x)dx + \sum_{j=1}^{N} K_{TMD\ J}(\omega_n)\psi_n(x_j)\psi_m(x_j) = 0$$

$$\int_0^L \frac{d^2\psi_m(x)}{dx^2}\frac{d^2\psi_n(x)}{dx^2}dx - \sigma_m{}^2\int_0^L \psi_m(x)\psi_n(x)dx + \sum_{j=1}^{N} K_{TMD\ J}(\omega_m)\psi_m(x_j)\psi_n(x_j) = 0$$

Integrating by parts and subtracting the first equation from the second yields the first orthogonality condition

$$(\sigma_m{}^2 - \sigma_n{}^2)\int_0^L \psi_n(x)\psi_m(x)dx + \sum_{j=1}^{N}[K_{TMD\ J}(\omega_n) - K_{TMD\ J}(\omega_m)]\psi_n(x_j)\psi_m(x_j) = 0$$

Multiplying the first two equations by σ_n and σ_m respectively and subtracting the first from the second, leads to the second orthogonality condition:

$$(\sigma_m - \sigma_n)\int_0^L \frac{\partial^2\psi_m(x)}{\partial x^2}\frac{\partial^2\psi_n(x)}{\partial x^2}dx - \sigma_m\sigma_n(\sigma_m - \sigma_n)\int_0^L \psi_n(x)\psi_m(x)dx + [\sigma_m K_{TMD\ J}(\omega_n) - \sigma_n K_{TMD\ J}(\omega_m)]\psi_n(x_j)\psi_m(x_j) = 0$$

MULTI-SPAN, RESPONSE TO A FORCING ACTION CONTINUED

Following this the impulse response function is derived, leading to:

$$w(x,t)=\sum_{k=1}^{\infty}\psi_k(x)\,\frac{2}{\Xi_k\,\omega_k}\int_0^t f(\tau)\,Sin\left(\omega_k(\tau-t)\right)d\tau$$

Where: $$\Xi_k = 2\int_0^L \bar{m}(x)\left[\psi_k(x)\right]^2 dx+\sum_{j=1}^{N} TMD_j\,\left[\psi_k(x_j)\right]^2$$

Where:

$$\mathrm{TMD_j}=\left(\frac{2\,\beta\,\mathrm{M}_{TMD\,\mathrm{j}}\,\omega_k\,\mathrm{Cos}(2\alpha)+\sqrt{\beta}\,\mathrm{Sin}(2\alpha)(-\beta+\omega_k{}^2\mathrm{M}_{TMD\,\mathrm{j}}{}^2)-2\,\rho_{spring\,\mathrm{j}}\,\omega_k(EA_{spring\,\mathrm{j}}\,\mathrm{M}_{TMD\,\mathrm{j}}+h_j(\beta+\omega_k{}^2\mathrm{M}_{TMD\,\mathrm{j}}{}^2))}{2\,\omega_k\left(\sqrt{\beta}\,\mathrm{Cos}(\alpha)-\omega_k\,\mathrm{Sin}(\alpha)\mathrm{M}_{TMD\,\mathrm{j}}\right)^2}\right)$$

And:

$$\alpha=\left(\frac{h_j\omega_k\sqrt{\rho_{spring\,\mathrm{j}}}}{\sqrt{EA_{spring\,\mathrm{j}}}}\right) \qquad \beta = EA_{spring\,\mathrm{j}}\,\rho_{spring\,\mathrm{j}}$$

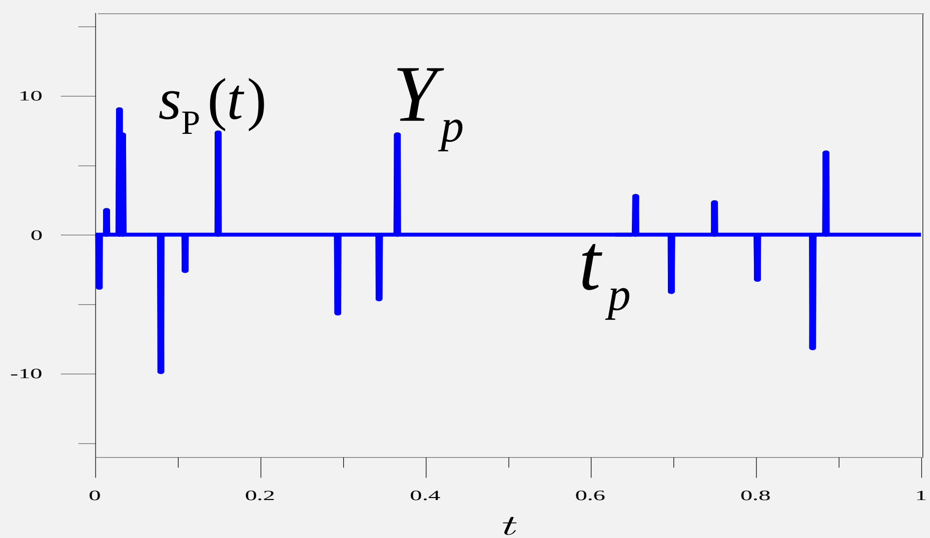
Considering a uniform moving load for k modes, this takes the form:

$$w(x,t)=\sum_{k=1}^{\infty}\left(\psi_k(x)\int_0^t e^{i\omega_k(t-\tau)}d\tau\right)\frac{\int_0^L \psi_k(x)\delta(x-V_0\,\tau)\,dx}{i\,\Xi_k\,\omega_k}$$

POISSONIAN LOADING: AN EXTENSION TO THE PROPOSED METHOD

- Poissonian loading is a special case of random loading in which loads of random amplitude arrive at independent random times. Therefore, random Poissonian loading can be described as a method of discretising truly random white noise.

- Poissonian processes are used extensively to model traffic loading, in our case, the particular process is a stationary poisson process. Here each load is assumed to be moving in the same direction at a constant and equal velocity, arriving at random times and with random magnitudes.

- 

Counting Process

$$s_P(t) = \sum_{p=1}^{N(t)} Y_p \delta(t - t_p)$$

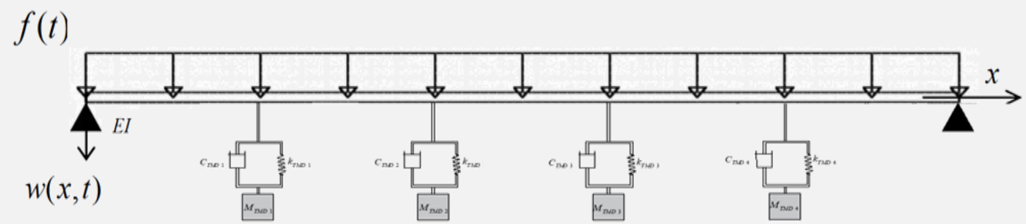
Random Variables

Poisson Distributed

APPLICATION TO THE MULTI-SPAN BEAM

- In the Equation of motion:

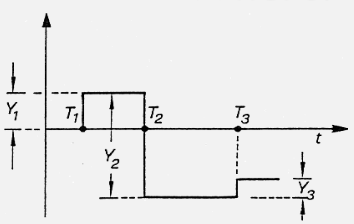
$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\bar{m}\frac{\partial^2 w(x,t)}{\partial t^2}+R(x,t)=f(t)$$

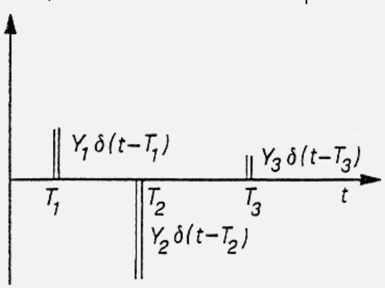


f(t) is expressed as:

$$\sum_{p=1}^{N(t)} Y_p\,\delta\left[x-\left(t-t_p\right)v\right]W\left(t-t_p,t_L\right)$$

Where:





Y_p Is the independent variable for the amplitude of the forcing actions

t_p Random variable (independent of Y) of arrival times distributed according to the Poisson Law

$N(t)$ Is a counting function

$\delta\left[x-\left(t-t_p\right)v\right]$ Is a Dirac's delta function denoting the position of the force where v is the velocity and t_p is the random time instant at which the force began

$W\left(t-t_p,t_L\right)$ Is a window function removing the force after it has traversed the beam

APPLICATION TO THE MULTI-SPAN BEAM

- This leads to the equation below

$$\ddot{g}_k(t) + \omega_k^{\,2} g_k(t) = \frac{2}{\Xi_k} s_k(t)$$

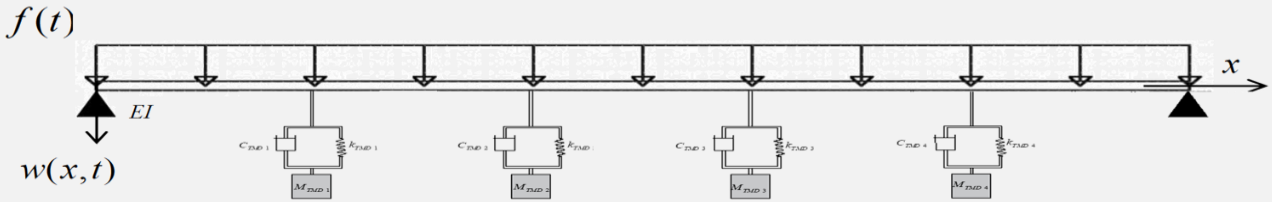
Where, as before:

$$\Xi_k = 2\int_0^L \bar{m}(x)\left[\psi_k(x)\right]^2 dx + \sum_{j=1}^{N} TMD_j \left[\psi_k(x_j)\right]^2$$

And $s_k(t)$ is the Poissonian load:

$$s_k(t) = \sum_{p=1}^{N(t)} Y_p \phi_k\left(t - t_p\right)$$

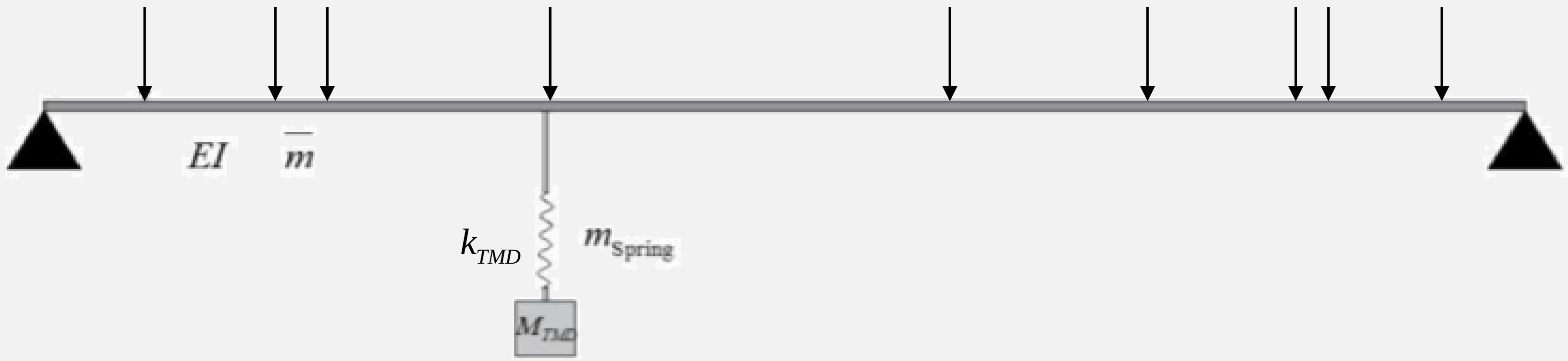
Where: $\phi_k\left(t - t_p\right) = \psi_k(v\tau) W(\tau, t_L)$



Ricciardi, G., 1994. Random Vibration of Beam under Moving Loads. *Journal of Engineering Mechanics*, 120(11), pp. 2361-2380.

NUMERICAL EXAMPLE

- The beam shown below is subjected to a series of random moving loads



$L = 6$ m
$E = 210$ GPa
$I = 0.000326\ m^4$
$\overline{m} = 487.5$ kg
$\rho = 7800\ kg/m^3$
$k_{TMD} = 13.125$ MN/m

$M_{TMD} = 500$ kg
$m_{Spring} = 312.5$ kg

$s_n(\tau) =$ Poissonian distribution between 40 kN and 240 kN
$v = 34$ m/s
$\lambda = 0.375$

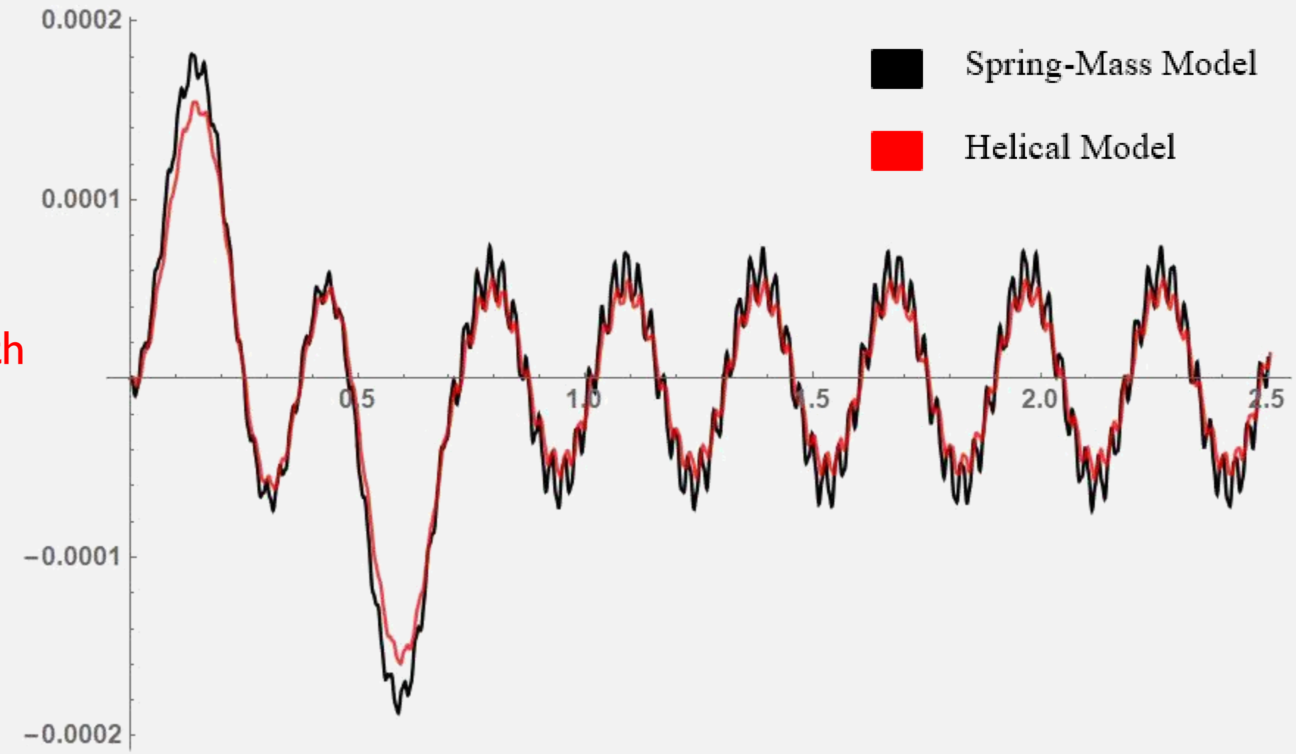
A Monte-Carlo simulation is then initiated to find the beam's response to a series of truly random loads. The simulation is run for 2000 samples (per mode) and the results are plotted in the following slides

NUMERICAL RESULTS

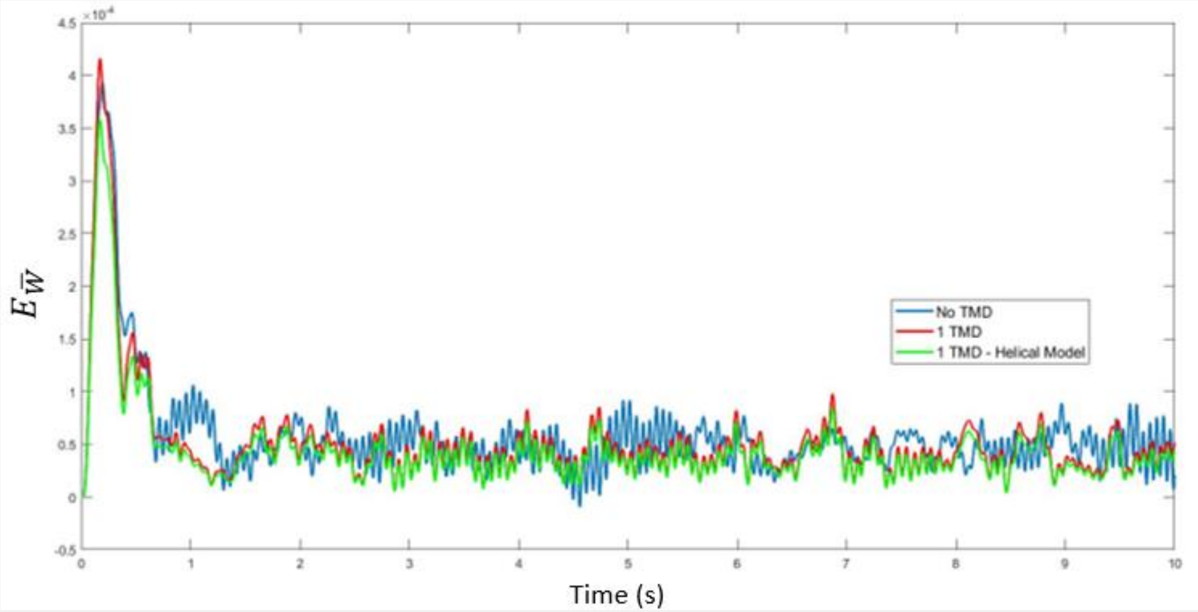
Comparison of natural frequencies obtained by the finite element method (FEM) and two approaches using the novel mathematical method. Note that the FEM model assumes that the spring element has a mass.

Natural Frequency	FEM	Helical Spring Model	Spring-Mass Model
1st	85.978 Rad/s	85.978 Rad/s	85.976 Rad/s
2nd	360.889 Rad/s	360.888 Rad/s	360.810 Rad/s
3rd	923.942 Rad/s	923.956 Rad/s	923.956 Rad/s
4th	1453.414 Rad/s	1453.410 Rad/s	1449.149 Rad/s
5th	2351.407 Rad/s	2351.838 Rad/s	2343.061 Rad/s

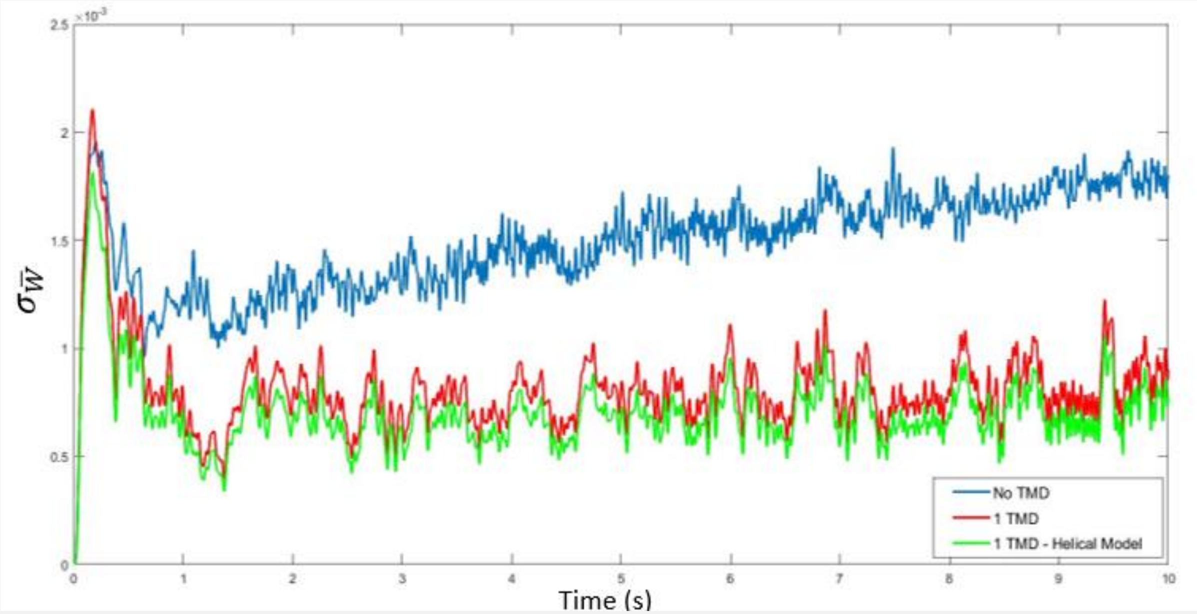
Comparison of the dynamic responses obtained from both mathematical models subjected to a moving load forcing action.



NUMERICAL RESULTS



Mean of the response to a series of random loads, also reported here is the response of a beam without an attached TMD



Standard deviation of the response to a series of random loads, following a Possonian distribution

CONCLUSIONS

- A novel method has been presented to calculate the dynamic response to deterministic loads and moving random loads of beams with any number of discontinuities, this novel method has also been extended to consider the Helical Spring model in which the spring element of the TMD has a mass and, thus, inertial effects.

- The helical model has been considered in an attempt to better model the real-world effects of a TMD, the results however call into question whether or not this extra complexity and computational effort is worth it as the standard model in which the spring element is assumed to be massless, performs similarly in both a deterministic and stochastic setting. As shown in the numerical application where a Monte-Carlo simulation was run.

- A formulation of the equation of motion within the theory of generalised functions has been proposed. The response has been calculated via modal superposition, deriving:
  - Novel closed-form expressions for the eigenfunctions by Fourier Transform of the equation of motion
  - Characteristic equation as determinant of a 4×4 matrix for any n. of attachments
  - Closed-form expressions for Duhamel integrals of m^{th} mode

- The proposed method yields significant computational advantages w.r.t. the classical modal superposition method which is based on dividing the beam into uniform segments between consecutive attachments, where the characteristic equation is built as determinant of a $4(N+1)\times4(N+1)$ matrix for N attachments.

- In the future we plan to validate these results by experimental testing particularly with respect to the difference between the helical and traditional TMD models

THANK YOU FOR YOUR TIME AND ATTENTION!

Contact Details

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