An efficient approach for reliability-based optimization of linear dynamical structures subject to Gaussian excitation

REC 2021

Danko Jerez

Institute for Risk and Reliability
Leibniz Universität Hannover

danko.jerez@irz.uni-hannover.de

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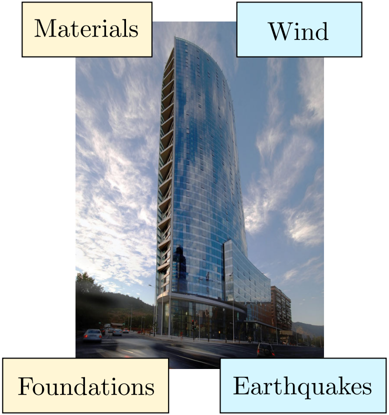
Introduction

- Structures must be designed in order to satisfy certain **design requirements**.
- **Uncertainties** must be considered in the design process.
  - **Stochastic excitations**.
- **Reliability-based design optimization** (RBO).



Reliability-based optimization (RBO) problem

$$
\begin{array}{llll}
\min\limits_{\mathbf{x}} & c(\mathbf{x}) & & \text{(cost function)} \\
\text{s.t.} & r_j(\mathbf{x}) \leq 0, & j = 1, \ldots, n_r & \text{(reliability constraints)} \\
 & g_j(\mathbf{x}) \leq 0, & j = 1, \ldots, n_g & \text{(standard constraints)} \\
 & x_i^L \leq x_i \leq x_i^U, & i = 1, \ldots, n_x & \text{(side constraints)}
\end{array}
$$

- Reliability constraints:

$$
r_j(\mathbf{x}) = P_{F_j}(\mathbf{x}) - P_{F_j}^* \leq 0, \qquad j = 1, \ldots, n_r
$$

- **Structural systems under stochastic excitation**.
  - **First-passage failure probabilities.**

First-passage failure probabilities

- Failure event:

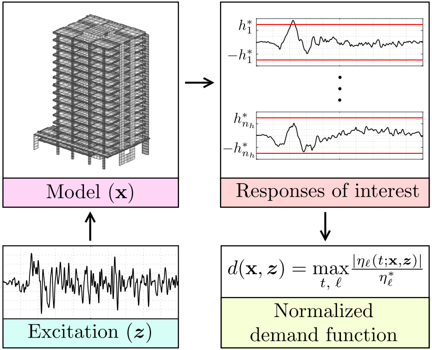
$$F = \{d(\mathbf{x}, \boldsymbol{z}) > 1\}$$

- Normalized demand function:

$$d(\mathbf{x}, \boldsymbol{z}) = \max_{\substack{\ell=1,\ldots,n_\eta \\ t\in[0,T]}} \frac{|\eta_\ell(t; \mathbf{x}, \boldsymbol{z})|}{\eta_\ell^*}$$

- First-passage failure probability:

$$P_F(\mathbf{x}) = \int I_F(\mathbf{x}, \boldsymbol{z}) f_{\boldsymbol{Z}}(\boldsymbol{z}) d\boldsymbol{z}$$



$\mathbf{x}$: design variables.
$\boldsymbol{z}$: basic random variables.

Linear structural systems

$$\boxed{\mathbf{M}\ddot{\mathbf{y}}(t,\mathbf{x},\boldsymbol{z}) + \mathbf{C}\dot{\mathbf{y}}(t,\mathbf{x},\boldsymbol{z}) + \mathbf{K}\mathbf{y}(t,\mathbf{x},\boldsymbol{z}) = \mathbf{q}\,p(t,\boldsymbol{z}), \quad t \in [0,T]}$$

- Karhunen-Loève expansion:

$$\boxed{p(t_k,\boldsymbol{z}) = \mu_k + \boldsymbol{\psi}_k^T \boldsymbol{z}, \quad k = 1,\ldots,n_T}$$

where $\boldsymbol{z} \in \mathbb{R}^{n_z}$ are standard Gaussian variables, $\boldsymbol{z} \sim f_{\boldsymbol{Z}}(\boldsymbol{z})$.

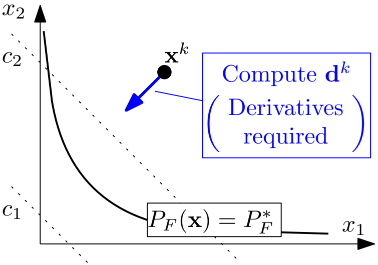
- Responses of interest are linear:

$$\boxed{\eta_\ell(t_k,\mathbf{x},\boldsymbol{z}) = \mathbf{a}_{\ell,k}^T(\mathbf{x})\boldsymbol{z}, \quad \ell = 1,\ldots,n_\eta,\; k = 1,\ldots,n_T}$$

Contents

Use of line search methods

1. Identify a search direction $\mathbf{v}^k$ at the current design $\mathbf{x}^k$ (**derivatives are required**).

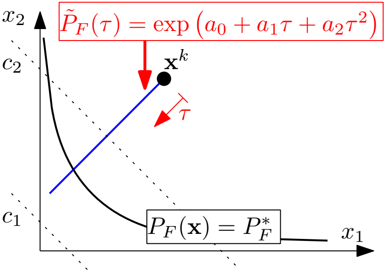


H.A. Jensen, L.G. Becerra & M.A. Valdebenito (2013). "On the use of a class of interior point algorithms in stochastic structural optimization", *Comput Struct* 126:69-85.

Use of line search methods

1. Identify a search direction $\mathbf{v}^k$ at the current design $\mathbf{x}^k$ (**derivatives are required**).

2. Formulate a **one-dimensional surrogate** $P_F(\mathbf{x}^k + \tau\mathbf{v}^k) \approx \tilde{P}_F(\tau)$ along $\mathbf{v}^k$.



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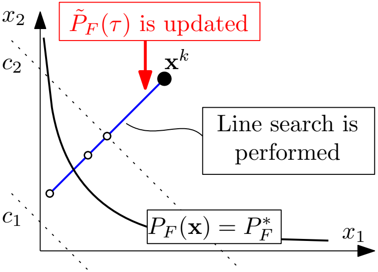

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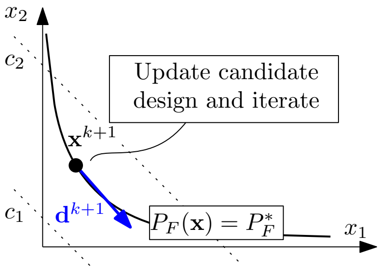
4. Update the current design.



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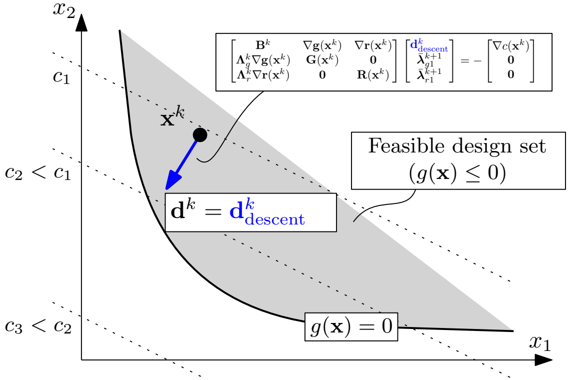
Feasible-descent direction $(\mathbf{d}^k)$



J. Herskovits & G. Santos (1997). "On the computer implementation of feasible direction interior point algorithms for nonlinear optimization", *Struct Optim* 14(2):165-172.

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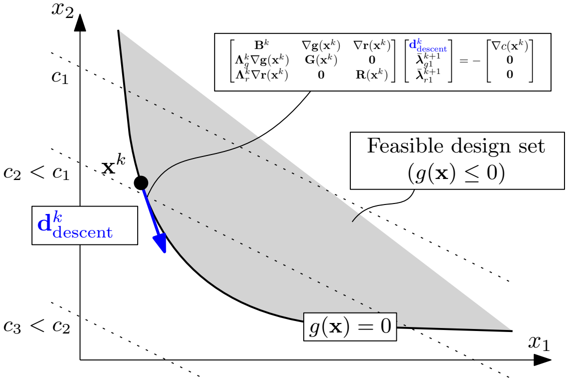

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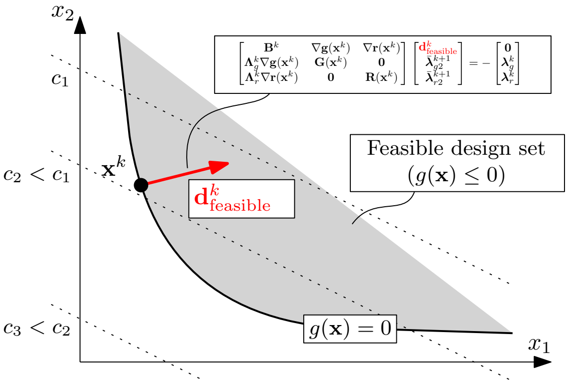

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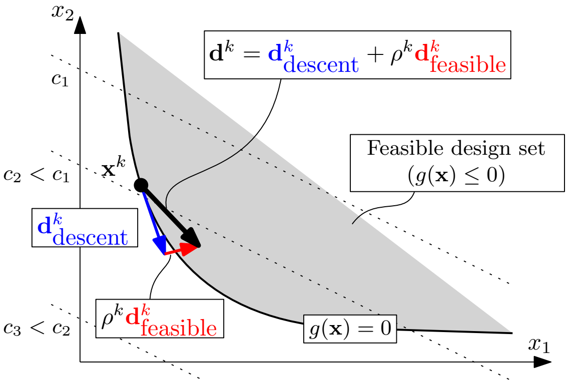

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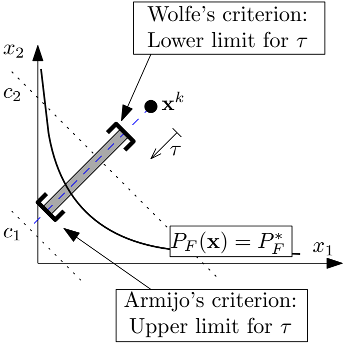
Inexact line search

Armijo's criterion:

$$\begin{cases} c(\mathbf{x}^k + \tau \mathbf{d}^k) \leq c(\mathbf{x}^k) + \tau \nu_1 \nabla c(\mathbf{x}^k)^T \mathbf{d}^k \\ \mathbf{x}^k + \tau \mathbf{d}^k \text{ is feasible.} \end{cases}$$

Wolfe's criterion:

$$\text{any} \begin{cases} \nabla c(\mathbf{x}^k + \tau \mathbf{d}^k)^T \mathbf{d}^k \geq \nu_2 \nabla c(\mathbf{x}^k)^T \mathbf{d}^k \\ g_j(\mathbf{x}^k + \tau \mathbf{d}^k) \geq \nu_2 g_j(\mathbf{x}^k) \\ r_j(\mathbf{x}^k + \tau \mathbf{d}^k) \geq \nu_2 r_j(\mathbf{x}^k) \end{cases}$$



J. Nocedal and S. J. Wright. *Numerical optimization*. Springer New York, 2006.

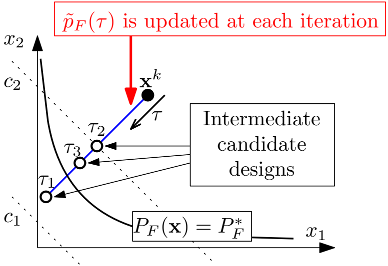
One-dimensional surrogate model

$$\ln P_F(\mathbf{x}^k + \tau \mathbf{d}^k) = p_F(\tau) \;\Rightarrow\; \boxed{\tilde{p}_F(\tau) = \alpha_0 + \alpha_1 \tau + \alpha_2 \tau^2, \quad \tau \geq 0}$$

Least-squares problem:

$$\min_{\alpha_0,\alpha_1,\alpha_2} \sum_{l=1}^{n_p} \left[w_1 \left[p_F(\tau_l) - \tilde{p}_F(\tau_l) \right]^2 + w_2 \left[p'_F(\tau_l) - \tilde{p}'_F(\tau_l) \right]^2 \right]$$

- **Reliability and directional derivative** values are considered.



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Directional Sampling

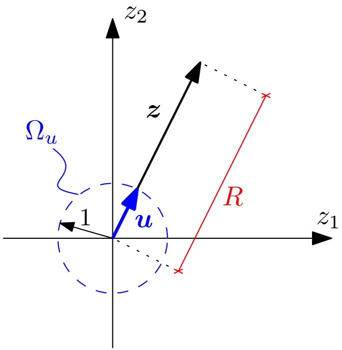
$$\boxed{\boldsymbol{z} = \|\boldsymbol{z}\| \frac{\boldsymbol{z}}{\|\boldsymbol{z}\|} = R\boldsymbol{u}}$$

- R^2: χ^2-distribution (n_z degrees of freedom).

$$R^2 \sim f_{R^2}(R^2) \;\Rightarrow\; R \sim 2R f_{R^2}(R^2)$$

- $\boldsymbol{u}$: uniform over Ω_u, where

$$\Omega_u = \{\boldsymbol{u} \in \mathbb{R}^{n_z} : \|\boldsymbol{u}\| = 1\}$$



$$\boxed{P_F = \int_{\Omega_u} \int_0^{\infty} I_F(R, \boldsymbol{u}) \left(2R f_{R^2}(R^2)\right) f_{\boldsymbol{U}}(\boldsymbol{u}) dR d\boldsymbol{u}}$$

Directional Importance Sampling (DIS)

Introducing a *directional importance sampling density*, $f_{\boldsymbol{U}}^{\text{IS}}(\boldsymbol{u})$:

$$\boxed{f_{\boldsymbol{U}}^{\text{IS}}(\boldsymbol{u}) = \sum_{i=1}^{n_\eta} \sum_{k=1}^{n_T} w_{i,k} f_{\boldsymbol{U}}(\boldsymbol{u}|F_{i,k})}$$

$$F_{i,k} = \underbrace{\left\{\mathbf{a}_{i,k}^T (R\boldsymbol{u}) > \eta_i^*\right\}}_{\substack{\text{Given response} \\ \text{and time instant}}}$$

$$\boxed{P_F^{\text{DIS}} = \int_{\Omega_u} \underbrace{\left[\int_0^\infty I_F(R,\boldsymbol{u}) \left(2R f_{R^2}(R^2)\right) f_{\boldsymbol{U}}(\boldsymbol{u}) dR \left(\frac{f_{\boldsymbol{U}}(\boldsymbol{u})}{f_{\boldsymbol{U}}^{\text{IS}}(\boldsymbol{u})}\right)\right]}_{\text{Computed in closed-form (linearity of the responses)}} f_{\boldsymbol{U}}^{\text{IS}}(\boldsymbol{u}) d\boldsymbol{u}}$$

M. Misraji, M. Valdebenito, H. Jensen & F. Mayorga. "*Application of directional importance sampling for estimation of first excursion probabilities of linear structural systems subject to stochastic Gaussian Loading*", Mech Syst Signal Process 139:106621 (2020).

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Directional Importance Sampling (DIS)

$$\boxed{\tilde{P}_F^{DIS} = \hat{P}_F \times \tilde{\kappa}}$$

Upper bound for P_F:

$$\boxed{\hat{P}_F = \sum_{i=1}^{n_\eta} \sum_{k=1}^{n_T} P\left[F_{i,k}\right]}$$

Correction factor (interaction between elementary failure domains):

$$\boxed{\tilde{\kappa} = \frac{1}{N} \sum_{j=1}^{N} \frac{1 - F_{R^2}\left(c(\boldsymbol{u}^{(j)})\right)}{\sum_{i=1}^{n_\eta} \sum_{k=1}^{n_T} \left[1 - F_{R^2}\left(c_{i,k}(\boldsymbol{u}^{(j)})\right)\right]}, \quad c_{i,k}(\boldsymbol{u}) = \frac{\eta_i^*}{|\mathbf{a}_{i,k}^T \boldsymbol{u}|}, \quad c(\boldsymbol{u}) = \min_{i,k} c_{i,k}(\boldsymbol{u})}$$

In general, a few hundred samples are required.

M. Misraji, M. Valdebenito, H. Jensen & F. Mayorga. "*Application of directional importance sampling for estimation of first excursion probabilities of linear structural systems subject to stochastic Gaussian Loading*", Mech Syst Signal Process 139:106621 (2020).

Reliability Sensitivity Estimates

$$\boxed{\frac{\partial \tilde{P}_F^{DIS}}{\partial x_i} = -\frac{1}{N}\sum_{j=1}^{N}\frac{2\hat{P}_F c(\mathbf{x},\boldsymbol{u}^{(j)})\frac{\partial c(\mathbf{x},\boldsymbol{u}^{(j)})}{\partial x_i} f_{R^2}(c(\mathbf{x},\boldsymbol{u}^{(j)})^2)}{\sum_{\ell=1}^{n_\eta}\sum_{m=1}^{n_T}(1-F_{R^2}(c_{\ell,m}(\mathbf{x},\boldsymbol{u}^{(j)})^2))}}$$

- Reliability sensitivities are computed as a **post-process** of the simulation results.
- One simulation run plus the sensitivities of the mode shapes and natural frequencies.

M. Valdebenito, M. Misraji, H. Jensen & F. Mayorga. "*Sensitivity estimation of first excursion probabilities of linear structures subject to stochastic Gaussian Loading*", Comput Struct 248:106482 (2021).

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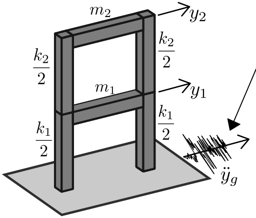
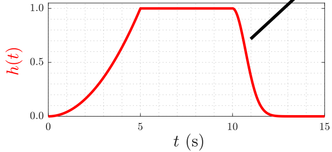
Two-degree-of-freedom system



$$\boxed{\ddot{y}_g(t) = \Omega_1^2\omega_1(t) + 2\xi_1\Omega_1\omega_2(t) - \Omega_2^2\omega_3(t) - 2\xi_2\Omega_2\omega_4(t)}$$

$$\begin{bmatrix} \dot{\omega}_1(t) \\ \dot{\omega}_2(t) \\ \dot{\omega}_3(t) \\ \dot{\omega}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\Omega_1^2 & -2\xi_1\Omega_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega_1^2 & 2\xi_1\Omega_1 & -\Omega_2^2 & -2\xi_2\Omega_2 \end{bmatrix} \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \\ \omega_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ h(t)w(t) \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} &m = 30 \times 10^3 \text{ kg}, \quad d = 5\% \\ &k_1 = \bar{k}\, x_1, \quad k_2 = \bar{k}\, x_2 \\ &\bar{k} = 18 \times 10^6 \text{ N/m} \end{aligned}}$$



- Modulated and filtered white noise excitation.
  - $\Omega_1 = 15.6$ rad/s, $\xi_1 = 0.6$, $\Omega_2 = 1$ rad/s, $\xi_2 = 0.9$, $S = 10^{-4}$ m^2/s^3.
- $T = 15$ s, $\Delta t = 0.01$ s $\Rightarrow$ $\boxed{n_z \approx 1500}$ (high-dimensional problem).

Design problem

Objective:

Minimize the **total stiffness** subject to a reliability constraint.

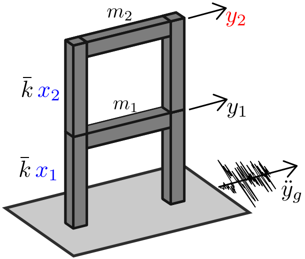
$$\min_{x_1, x_2} \quad c(\mathbf{x}) = x_1 + x_2$$
$$\text{s.t.} \quad P_F(\mathbf{x}) \leq 10^{-3}$$
$$0.5 \leq x_1, x_2 \leq 1.5$$

$$x_i = \frac{k_i}{\bar{k}}, \; i = 1, 2$$

Failure event: **Roof displacement**.

$$\boxed{d(\mathbf{x}, \boldsymbol{z}) = \max_{t \in [0,T]} \frac{|y_2(t, \mathbf{x}, \boldsymbol{z})|}{y_2^*}}$$

with $y_2^* = 0.006$ m.



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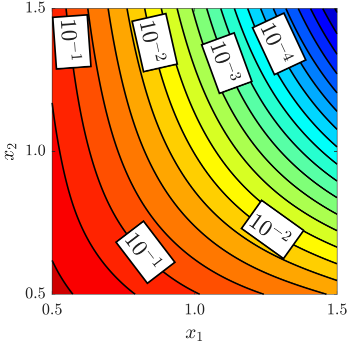
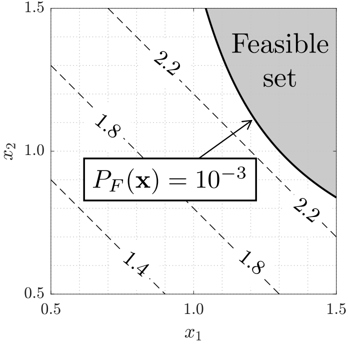
Design space

Contours of $P_F(\mathbf{x})$

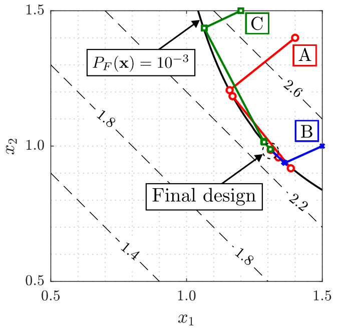


Sketch of the feasible set

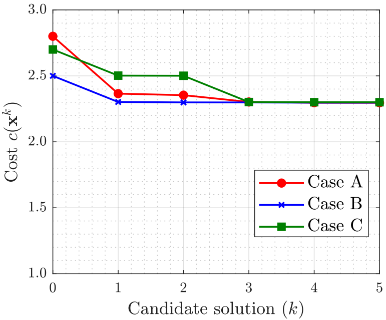


Results: Sequence of candidate designs

Evolution of candidate designs



Evolution of cost values

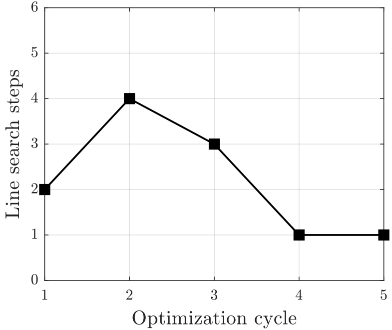


Final solution:

$$x_1^* = 1.31,\ x_2^* = 0.98$$
$$c(\mathbf{x}^*) = 2.29,\ P_F(\mathbf{x}^*) \approx 10^{-3}$$

Results: Metamodel performance

Failure probability evaluations during each optimization cycle (case A):



Generally, one to five iterations are required to find a new design.

Contents

Final remarks

- A sequential optimization strategy for the RBO of linear structures under Gaussian excitation has been presented.
- Failure probability functions are approximated along one dimension.
- Directional Importance Sampling is integrated for reliability and sensitivity assessment.

Future work:

- Integration of schemes to approximate directional derivatives.
- Implementation of reduced-order models and parametrization schemes.

Acknowledgements

The research reported here was supported in part by ANID (National Agency for Research and Development, Chile) under its program FONDECYT, grant numbers 1200087 and 1180271, and by ANID and DAAD (German Academic Exchange Service) under CONICYT-PFCHA/Doctorado Acuerdo Bilateral DAAD Becas Chile/2018-62180007. These supports are gratefully acknowledged by the authors.

Thank you for your attention.

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danko.jerez@irz.uni-hannover.de