Distribution-free P-box processes: definition and simulation

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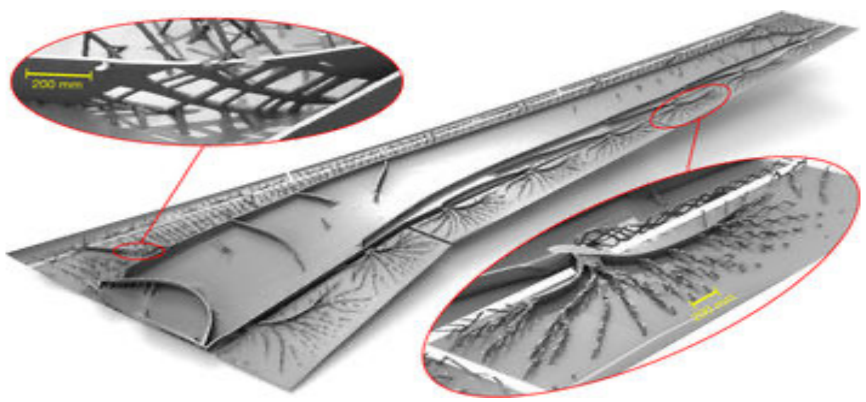
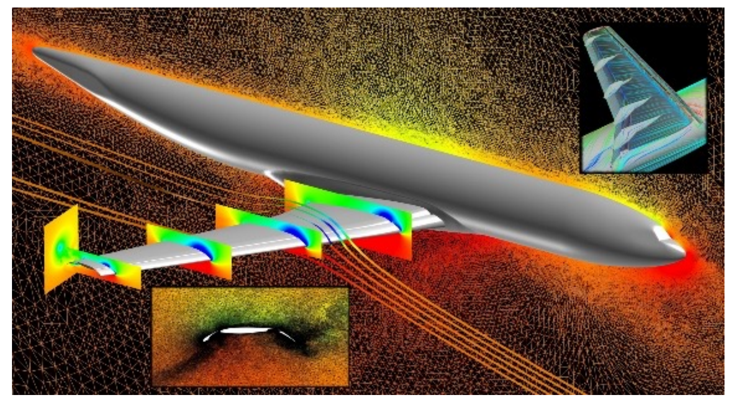
Introduction

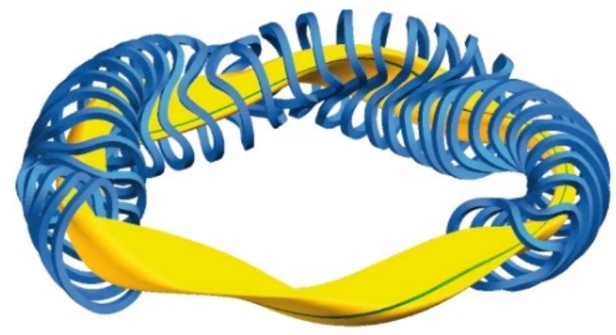
modern design methods allow for assessment of structural quality and thorough design optimization long before first part has been produced



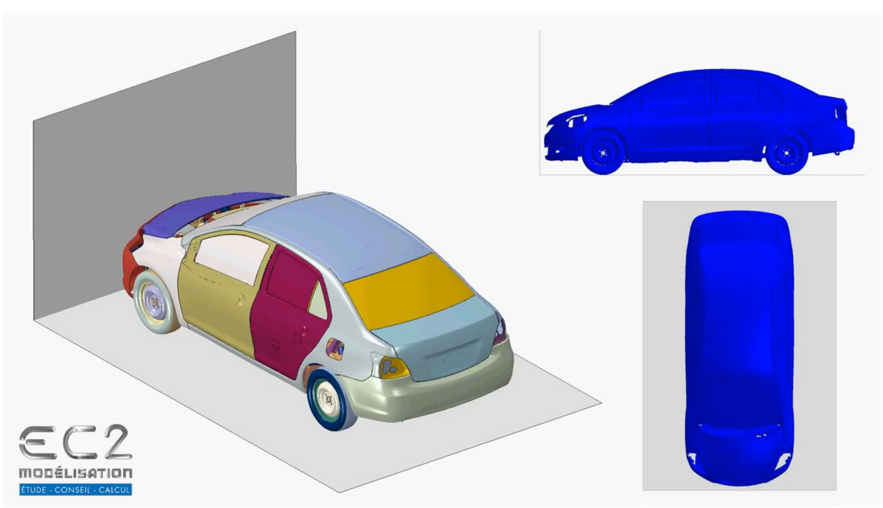
Topology optimised Airplane wing
(Nature)



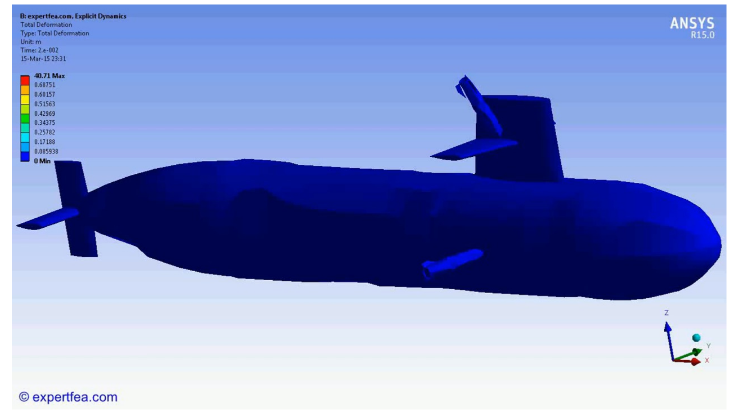
Airflow around Boeing 737 body
(NASA / Boeing)



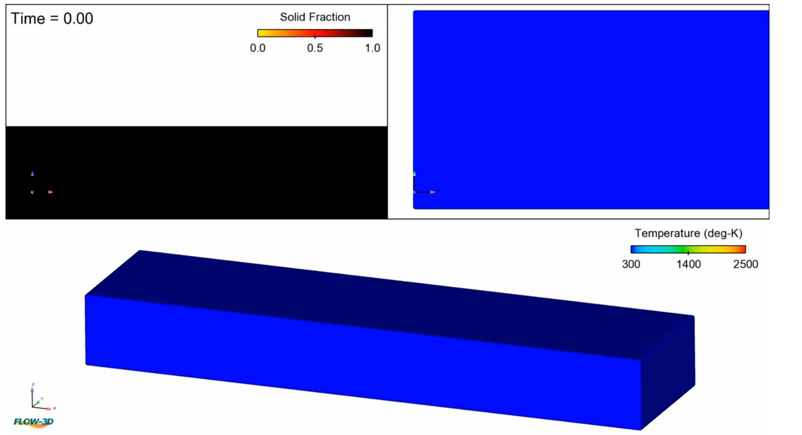
Wendelstein 7-X fusion reactor
(Max Planck Institute)



Car crash simulation
(Toyota Yaris)



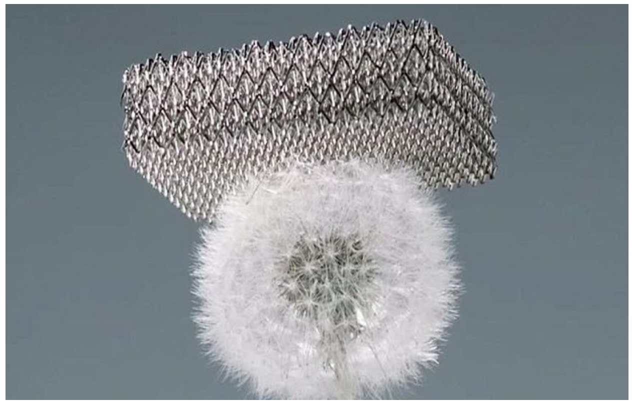
Dual torpedo impact
(Ansys)



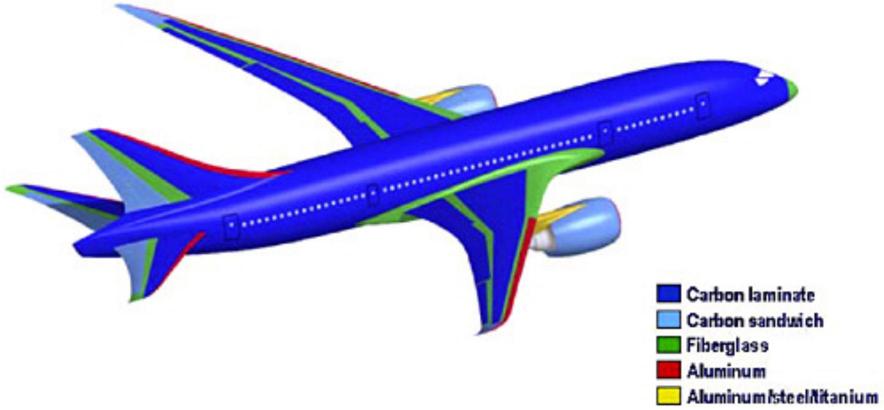
Laser Metal Deposition
(FlowScience)

Introduction

Advances in materials and manufacturing engineering allow for producing these highly optimized parts



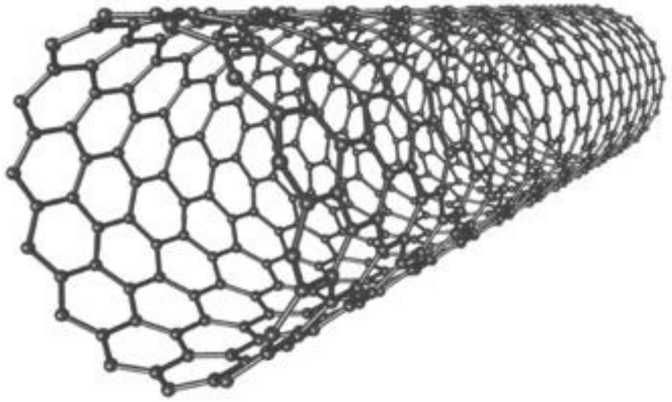
Micro lattice, produced via Additive Manufacturing (Boeing)



Composite material usage in Boeing 787 – Dreamliner (Boeing)

Additive Manufacturing inside the ISS (NASA)

Cranial implant via Additve Manufacturing (EOS)



Graphene nanotube (Harvard School of Sciences)

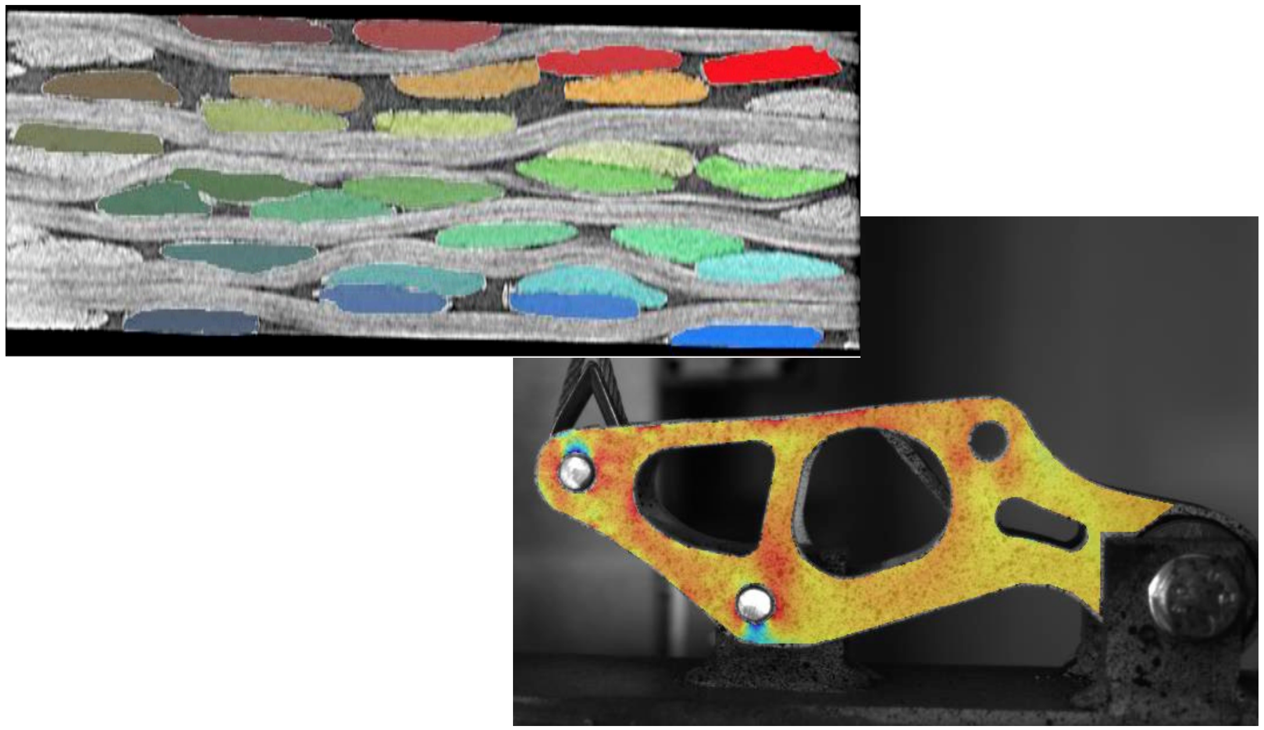
Multi-material Additive Manufacturing bionic arm (University of Nottingham)

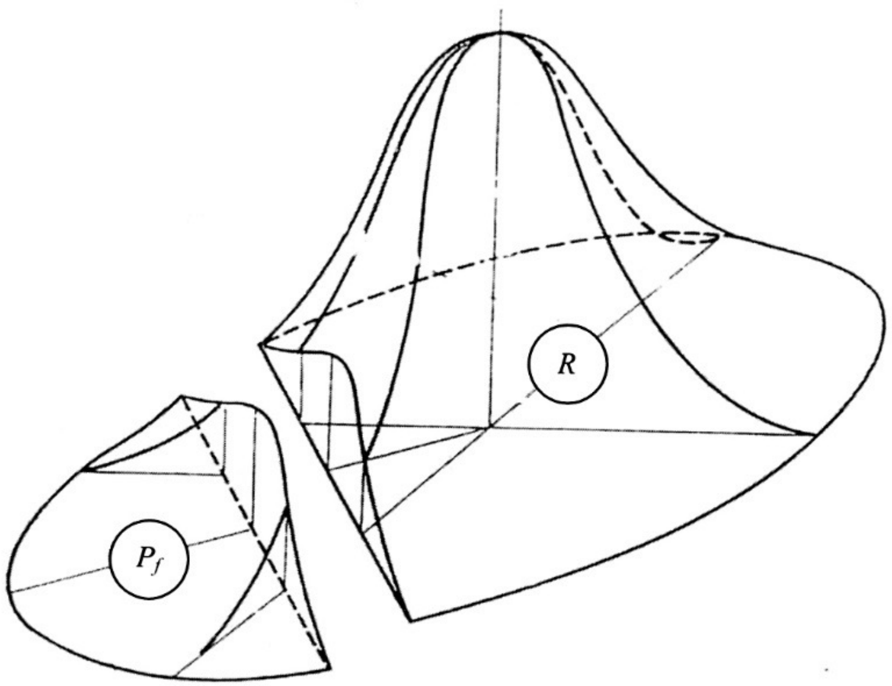
Reliability engineering

- Given these sources of uncertainty; how do we design safe and reliable components?

→ Probabilistic reliability analysis

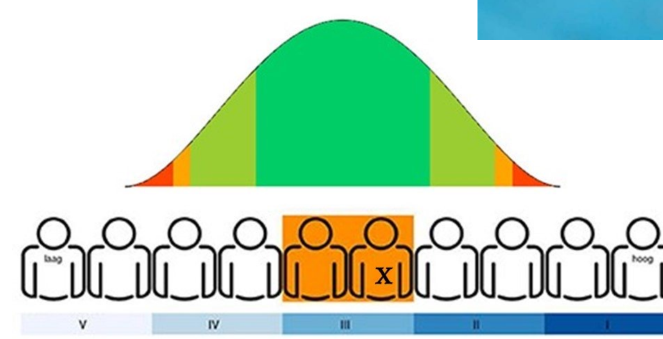
- Based on conceptions of probability theory
- Variability is modelled using probability density function
- Interested in probability that the structure fails

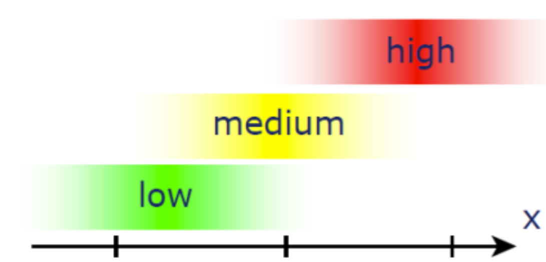


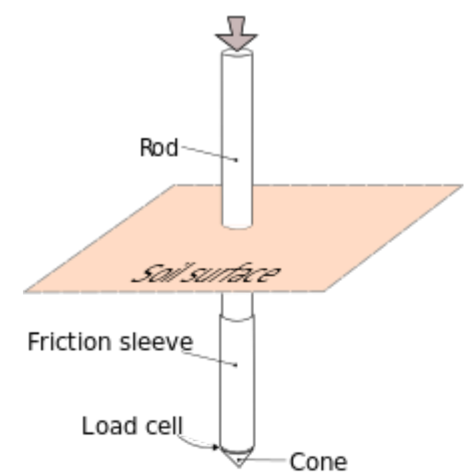


Reliability analysis with real data

- $f_X(\boldsymbol{x})$ needs to be estimated to perform reliability analysis
- However, estimation is complicated by:
  - Imprecise measurements
  - Small sample set sizes
  - Incomplete expert elicitations
  - Changing environmental conditions
  - Vague or dubious information
  - Expert assessment / experience
  - Linguistic assessments
  - Conditional probabilities observed under unclear conditions
  - Only marginals are available
  - …

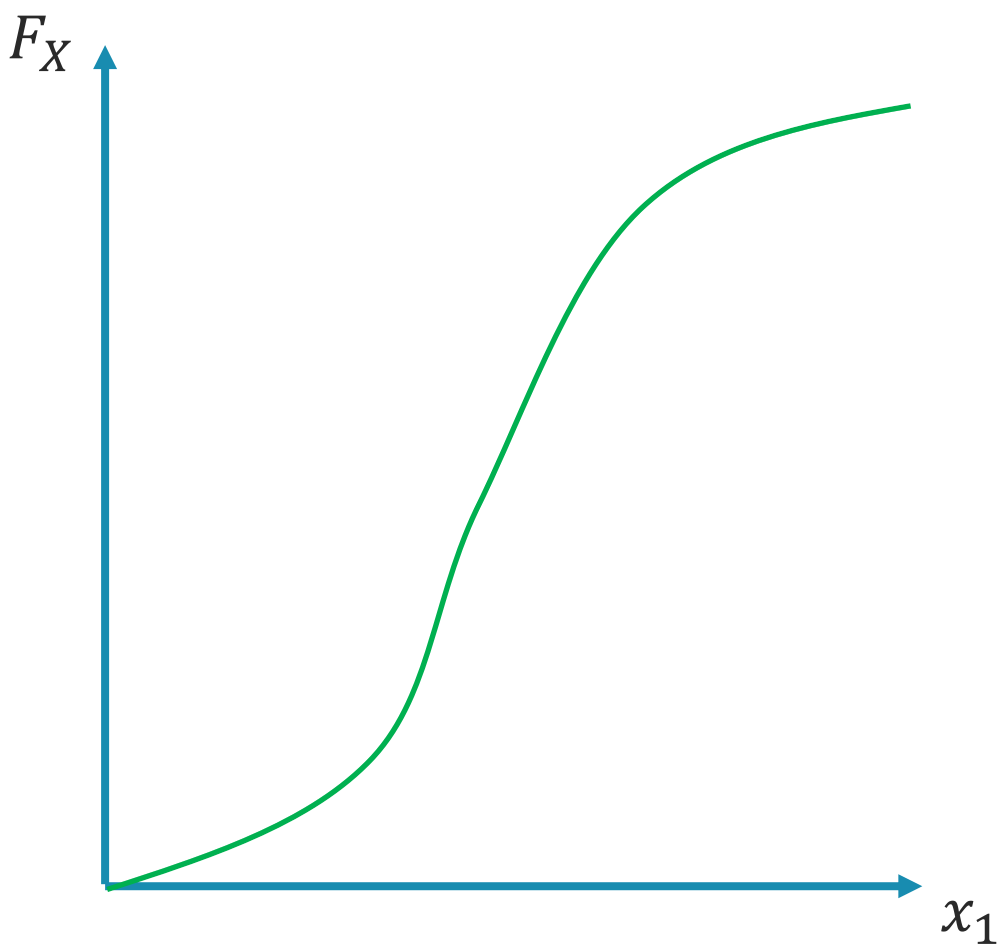


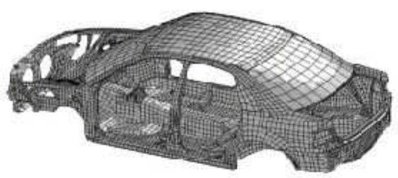




→ Mixtures of information from several sources with different levels of fidelity

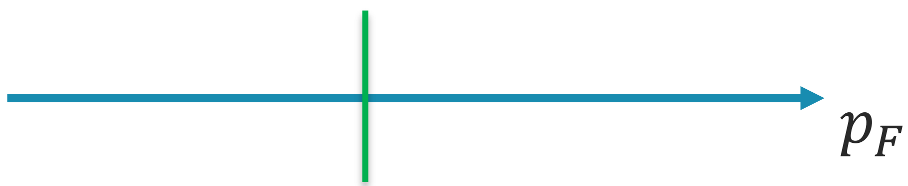
Reliability analysis



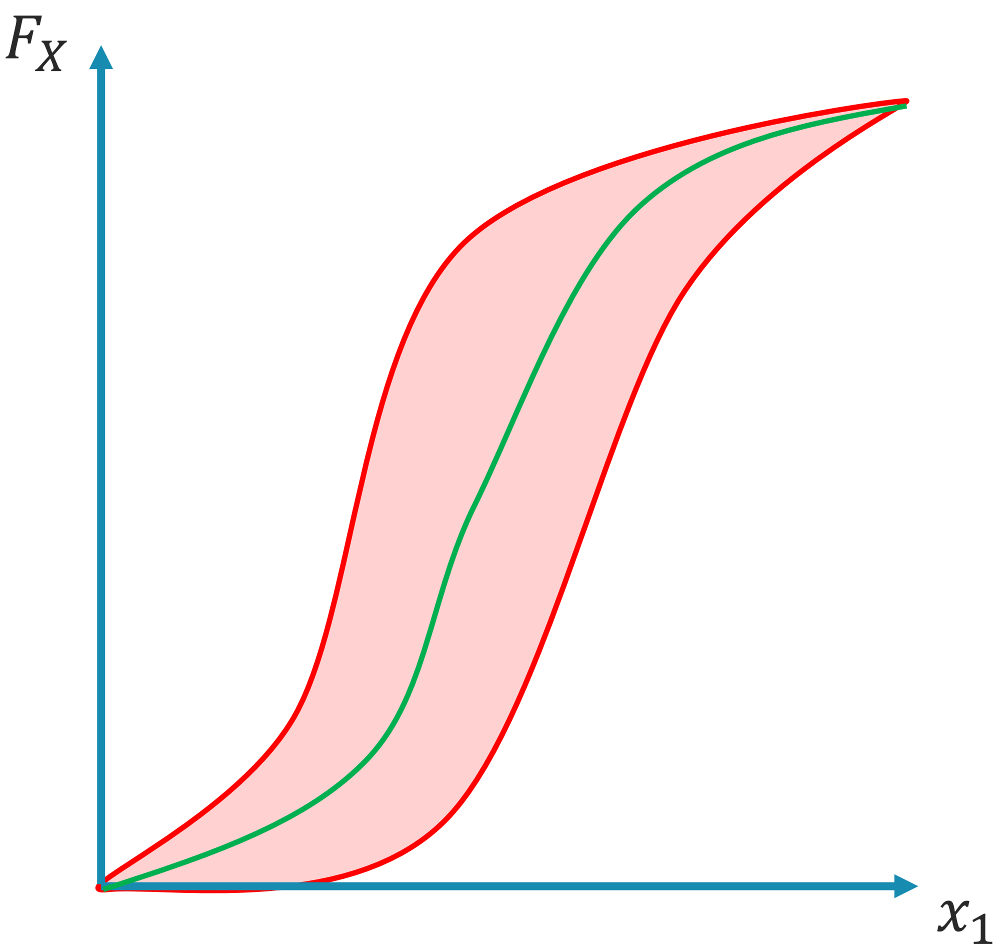


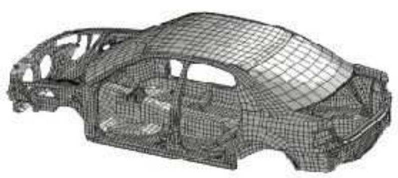
nominal model

$$p_F = \int_{D_x} \mathrm{I}_{\mathrm{M}}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}$$



Imprecise information

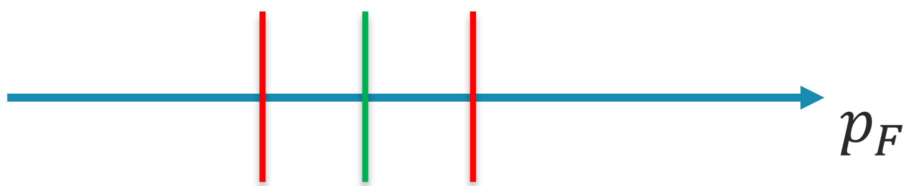




nominal model

$$\underline{p}_F = \min_{f_X} \int_{D_x} I_M(\boldsymbol{x}) f_{\boldsymbol{X}}(x) d\boldsymbol{x}$$

$$\overline{p}_F = \max_{f_X} \int_{D_x} I_M(\boldsymbol{x}) f_{\boldsymbol{X}}(x) d\boldsymbol{x}$$



Measure for “trustworthiness” of results

How to define this set of $f_X(x)$?

Probability boxes

Idea: provide set of possible distribution functions $F_{\boldsymbol{X}}(\boldsymbol{x})$ bounded by lower CDF $\underline{F}_{\boldsymbol{X}}(x) \in \mathbb{F}$ and upper CDF $\overline{F}_{\boldsymbol{X}}(\boldsymbol{x}) \in \mathbb{F}$, with $\mathbb{F}$ the set of all CDFs on $D_X \subseteq \mathbb{R}$

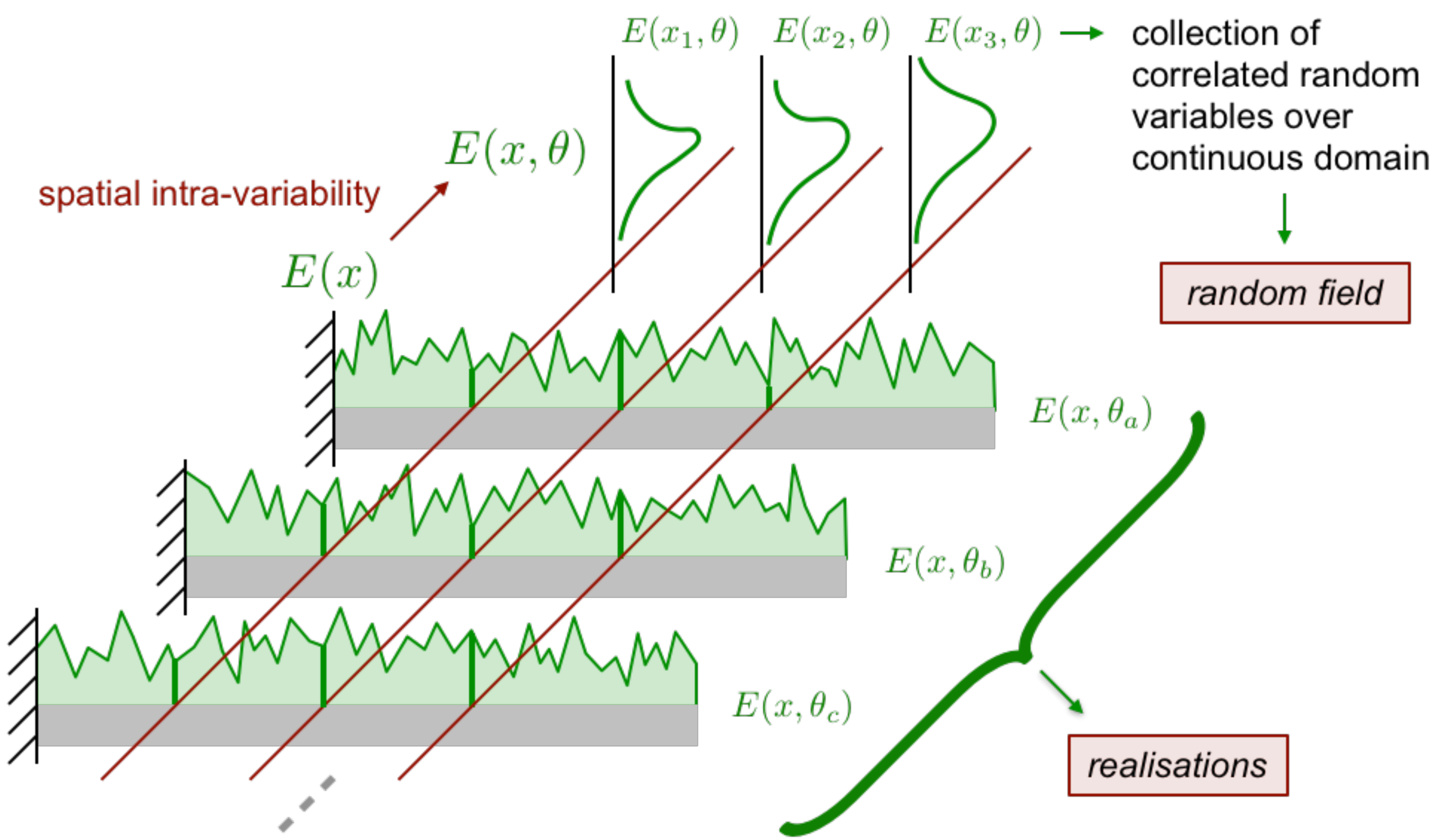
- Formally, a p-box is defined as the set $\{F_X(x) \in \mathbb{F} | \underline{F}_{\boldsymbol{X}}(x) \leq F_X(x) \leq \overline{F}_X(x), x \in D_X\}$
- Epistemic uncertainty on $F_X(x)$ is accounted for explicitly by assigning an interval $[\underline{F}_X(x), \overline{F}_X(x)]$ for each value of $x \in D_X$
- Small epistemic uncertainty: $[\underline{F}_X(x), \overline{F}_X(x)]$ is a tight interval

  → Large confidence in CDF and results
- Large epistemic uncertainty: $[\underline{F}_X(x), \overline{F}_X(x)]$ is a wide interval

  → Low confidence in CDF and results

  → collect more data

Modelling of space/time uncertainty



Modelling space/time uncertainty with p-boxes?

- Stochastic modelling: Karhunen-Loève expansion:

$$x(t,\xi) = \mu_x(t) + \sigma_x \sum_{i=1}^{\infty} \sqrt{(\lambda_i)}\, \psi_i(t)\xi$$

- Based on eigen-decomposition of autocorrelation function $R_{xx}(t,t')$:

$$\int_{\Omega} R_{xx}(t,t')\psi_i(t')dt' = \lambda_i \psi_i(t)$$

- In practice, KL expansion is truncated after T terms for computational reasons
- Remember; be careful with exponential kernels (see talk M. Broggi 18/05)

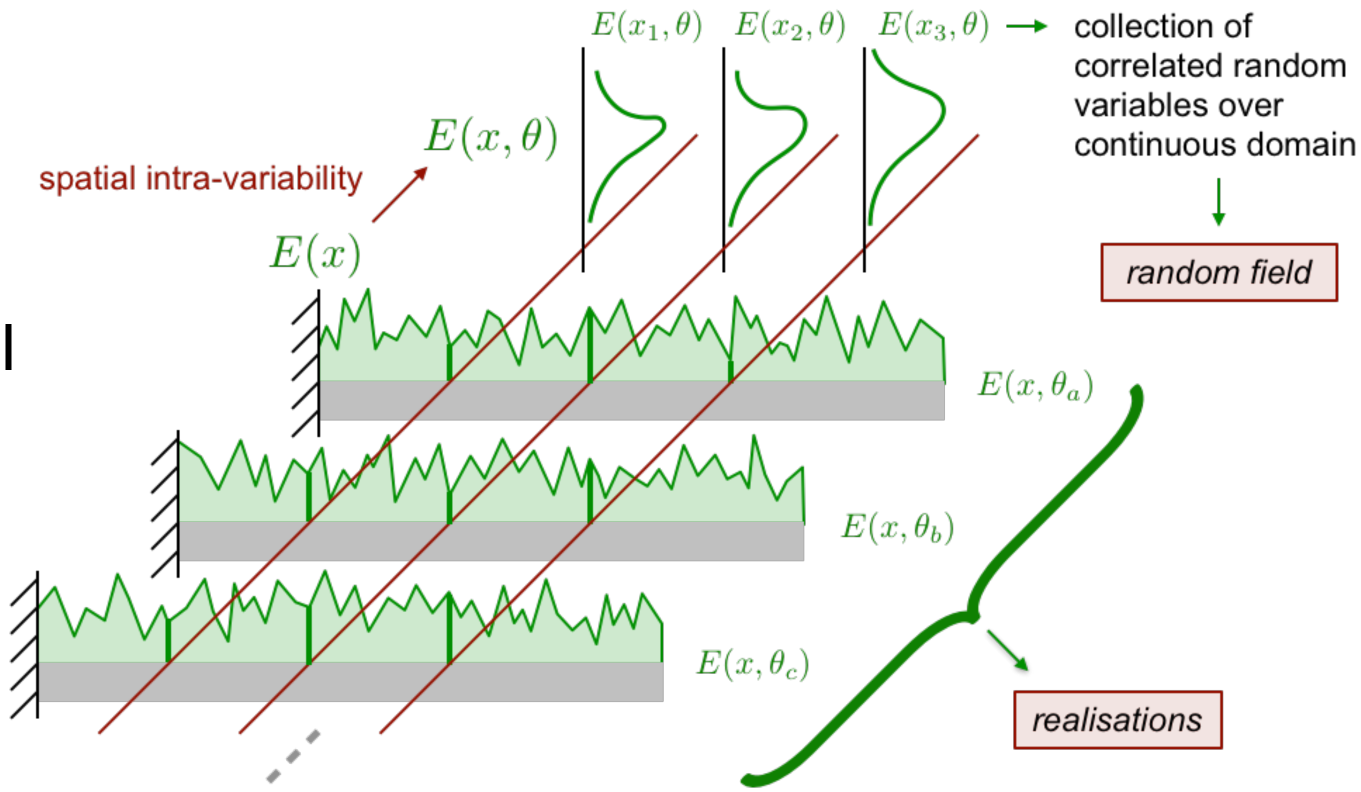


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Illustration: imprecise Clough-Penzien spectrum

- Example: the Clough-Penzien spectrum to represent earthquakes

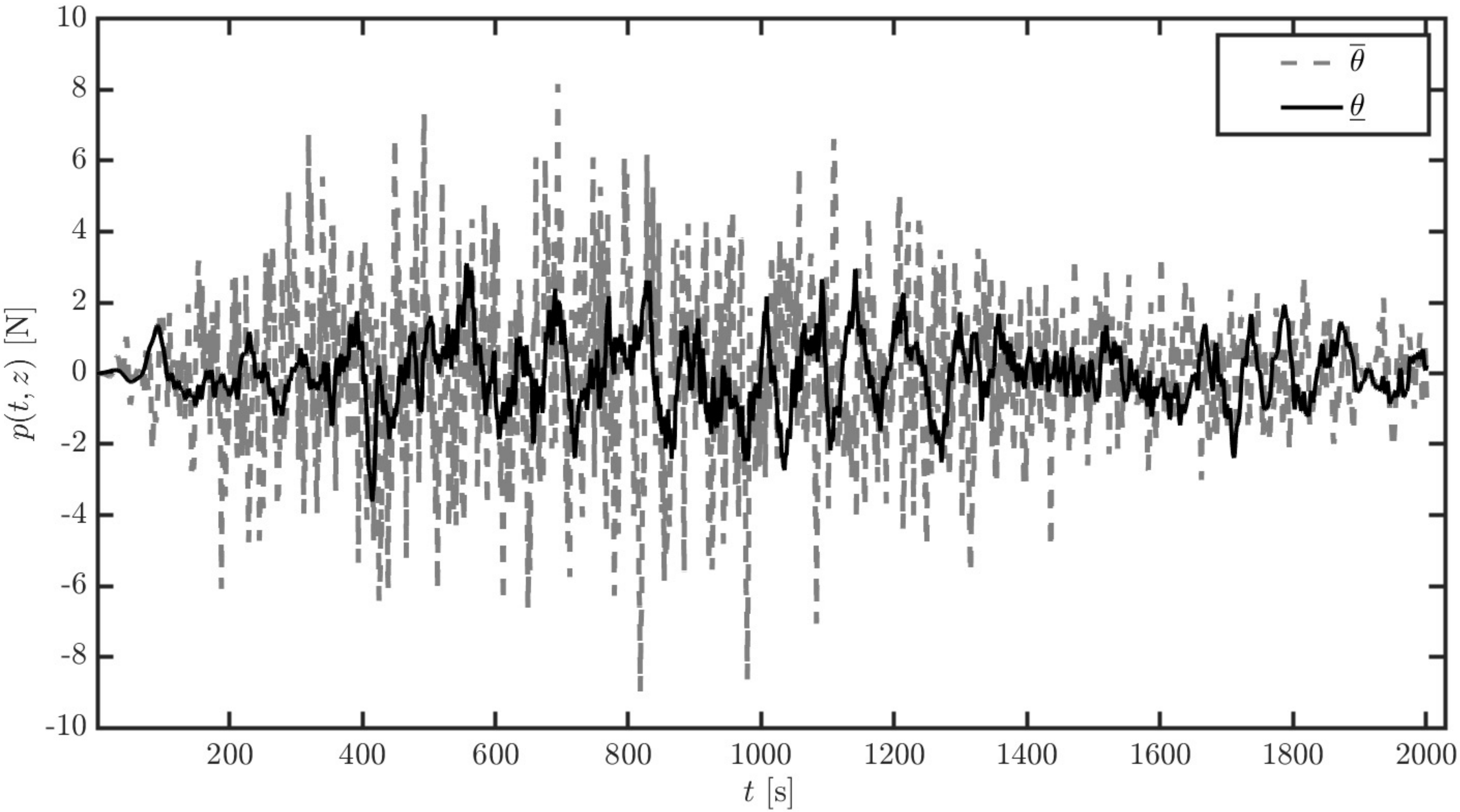
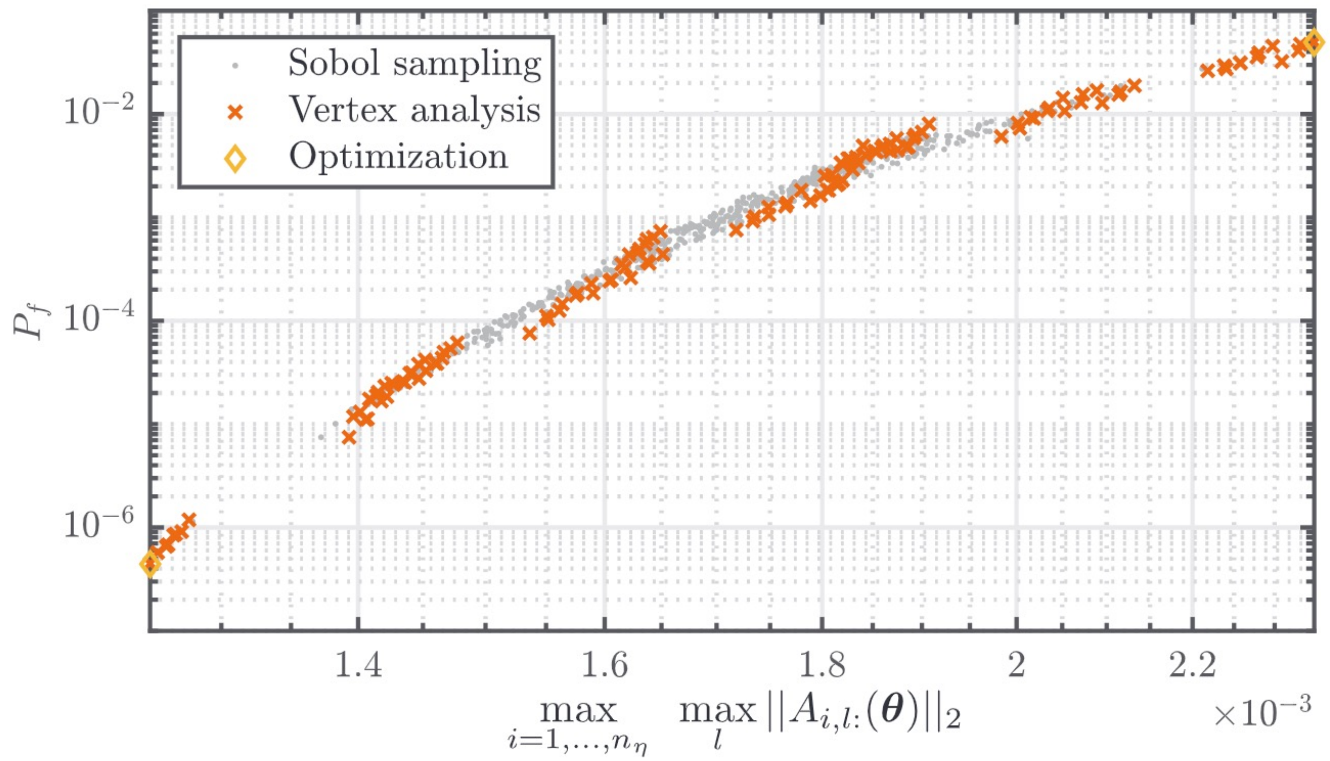
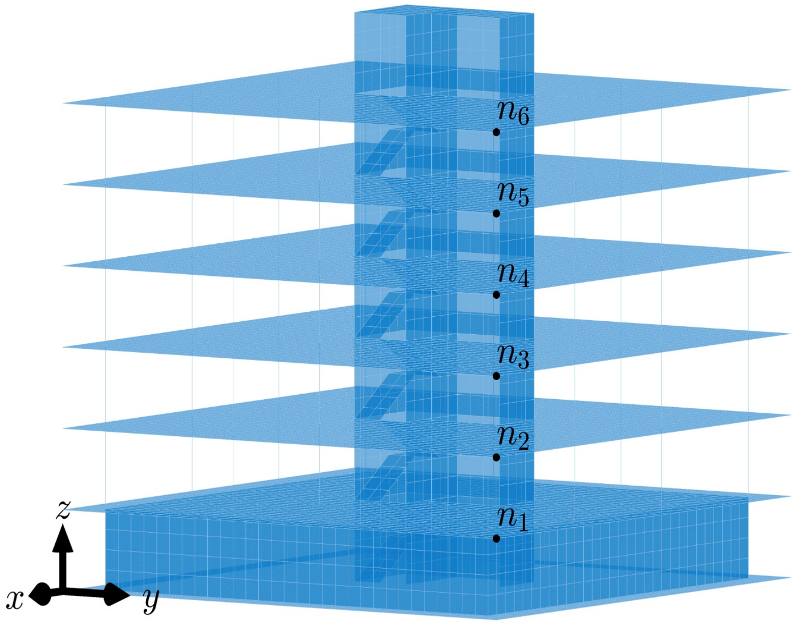


Illustration: imprecise Clough-Penzien spectrum

- 6 story building
  - Reinforced concrete
  - 9500 shell & beam elements
- QOI: inter-story drift
- Load: stochastic base excitation
  - Gaussian stochastic process
  - Autocorrelation governed by modulated Clough-Penzien spectrum with soil parameters between ‘firm’ and ‘medium’
- Probability of failure bounds span 6 orders of magnitude!

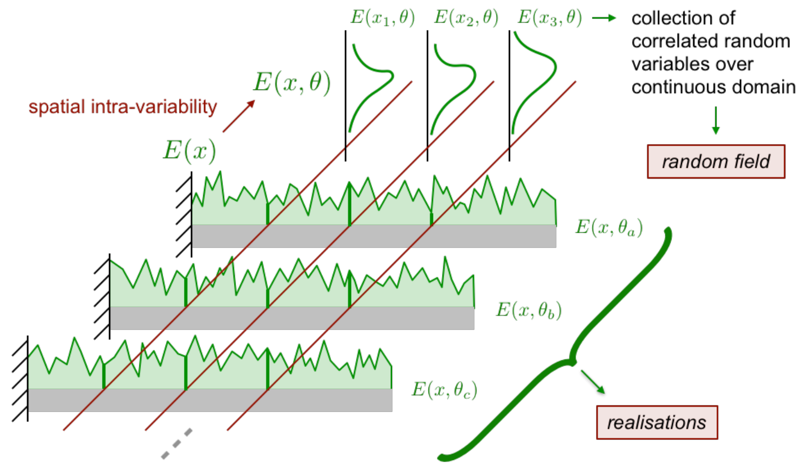
Faes, M. G. R., Valdebenito, M. A., Moens, D., & Beer, M. (2020). Bounding the first excursion probability of linear structures subjected to imprecise stochastic loading. *Computers & Structures*, *239*, 106320.





Distribution-free p-box processes

- Stochastic process $x(t,\xi)$: collection of correlated random variables $x(t_i,\xi)$ throughout Ω_d
  - o Distributed according to F_X
  - o Correlation governed by $R_{xx}(t,t')$



- What if we can't assume a single distribution F_X?

- P-box stochastic process $\hat{x}(t,\xi)$: collection of correlated p-boxes throughout Ω_d
  - o Distributed according to set $\left\{F_X(x) \in \mathbb{F} \middle| \underline{F}_X(x) \leq F_X(x) \leq \overline{F}_X(x), x \in D_X\right\}$
- Questions
  - o How do we deal with non-Gaussian realizations of the p-box?
  - o How to draw samples from this collection of correlated p-boxes?
  - o How to propagate this construction towards bounds on the reliability of the structure under consideration?

P-box valued stochastic processes

How do we deal with non-Gaussian realizations of the p-box?

- Translation theory:

$$x(t,\xi) = F_X^{-1} \circ \Phi\big(\eta(t,\xi)\big) = g\big(\eta(t,\xi)\big)$$

with $\eta(t,\xi)$ a standard Normal stochastic process, Φ the standard normal CDF and F_X^{-1} the inverse CDF of the target distribution

- Closed-form solutions exist for moments
- Extension towards free p-boxes:
  - Make translation map interval-valued:

$$\hat{x}(t,\xi) = \left[\underline{F}_X^{-1}, \overline{F}_X^{-1}\right] \circ \Phi\big(\eta(t,\xi)\big) = g^I\big(\eta(t,\xi)\big)$$

P-box valued stochastic processes

How to draw samples from this collection of correlated p-boxes?

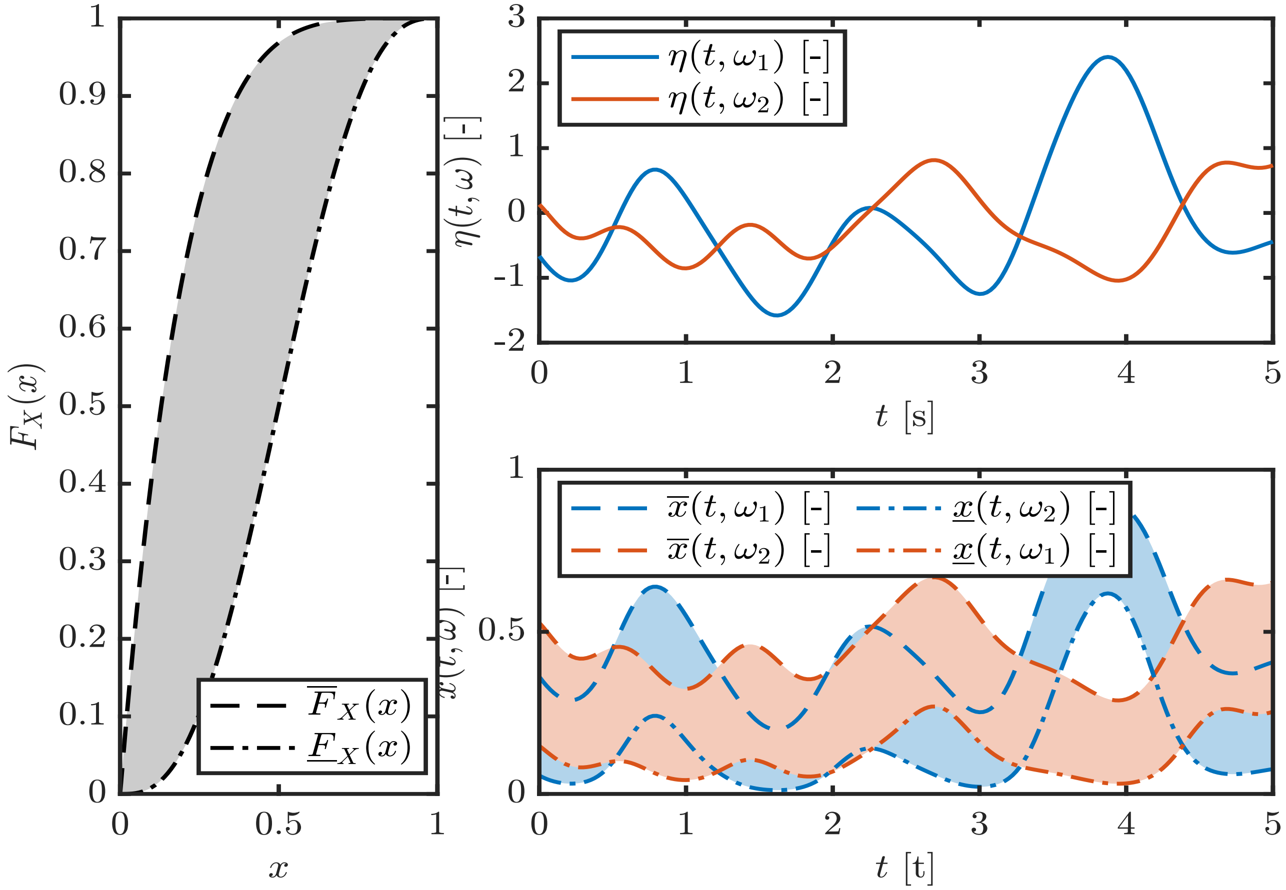
- Models a set of random fields that is consistent with F_X^I

- Each realization $F_X^k \in F_X^I$ yields a random field:

$x_k(t,\xi) = F_X^k \circ \Phi(\eta(t,\xi))$

- Each ξ_i yields an interval field:

$x^I(t,\xi_j) = \left[\underline{F}_X^{-1}, \overline{F}_X^{-1}\right] \circ \Phi\left(\eta(t,\xi_j)\right)$



P-box valued stochastic processes

How do we propagate these sets of fields towards bounds on the reliability of the structure under consideration?

- To compute the bounds on P_f, two optimization problems need to be solved:

$$\underline{P}_f = \min_{F_X^k \in F_X^I} P_f\left(F_X^k\right)$$
$$\overline{P}_f = \max_{F_X^k \in F_X^I} P_f\left(F_X^k\right)$$

- These optimization problems yield $F_{\underline{X}}^*$ and $F_{\overline{X}}^*$ for respectively lower and upper bound

- Infinite set of functions, which is computationally intractable

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P-box valued stochastic processes

How do we propagate these sets of fields towards bounds on the reliability of the structure under consideration?

- These continuous problems are replaced by discrete problems for computational tractability:

$$\underline{P_f} = \min_{\mathbf{F}_{\mathrm{X}}^{\mathrm{k}} \in \mathbf{F}_{\mathrm{X}}^{\mathrm{I}}} P_f\left(\mathbf{F}_{\mathrm{X}}^{\mathrm{k}}\right)$$
$$\overline{P}_f = \max_{\mathbf{F}_{\mathrm{X}}^{\mathrm{k}} \in \mathbf{F}_{\mathrm{X}}^{\mathrm{I}}} P_f\left(\mathbf{F}_{\mathrm{X}}^{\mathrm{k}}\right)$$
$$s.t.\ \boldsymbol{A}\mathbf{F}_{\mathrm{X}} \leq 0,$$

where $\mathbf{F}_{\mathrm{X}} = F_X(x_s)$ with x_s a vector of n_s evenly spaced sample points throughout the support of F_X^I

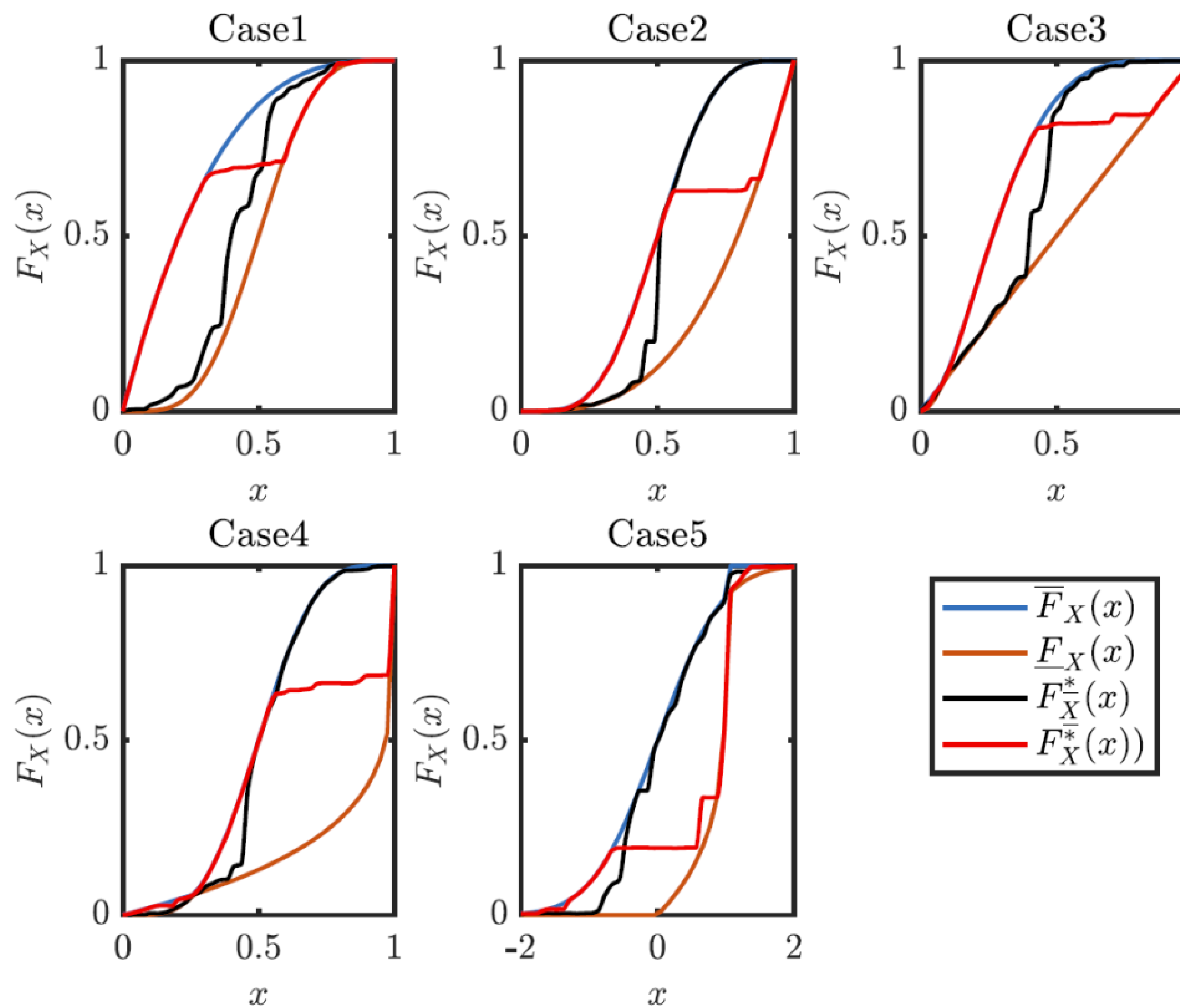
- Matrix $\boldsymbol{A}$ is included to ensure strict monotonicity of the resulting F_X^k.
- Inverse ${\mathbf{F}_{\mathrm{X}}^{\mathrm{k}}}^{-1}$ is calculated by means of piecewise cubic Hermite polynomial interpolation using Fritsch-Carlson algorithm

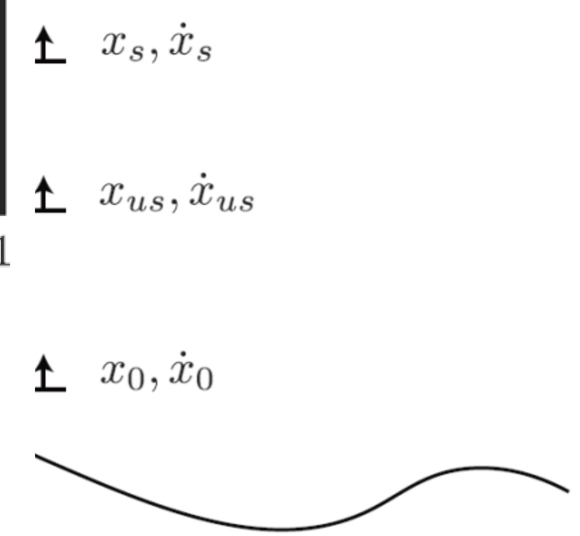
Example: car dynamics

- 2DOF oscill
- Base excitat ocess
- Compute fo
  - Failure = on accele

$\mathcal{M}(x) = 1 -$

- 5 case studi represented





Case	$\underline{F}_X(x)$		
1	$\min[\mathcal{B}(1,3), \mathcal{B}(5$		
2	$\min[\mathcal{B}(3,1), \mathcal{B}(5$		
3	$\min[\mathcal{B}(1,1), \mathcal{B}(2,5)])$	$\max[\mathcal{B}(1,1), \mathcal{B}(2,5)])$	0.5
4	$\min[\mathcal{B}(1,0.2), \mathcal{B}(5,5)])$	$\max[\mathcal{B}(1,.02), \mathcal{B}(5,5)])$	0.5
5	$\min[\mathcal{N}(0,0.75), \mathcal{B}(1,0.2)])$	$\max[\mathcal{N}(0,0.75), \mathcal{B}(1,0.2)])$	2

Open Issue

- The autocorrelation $R_{xx}(t,t')$ of a projection $x(t,\xi)$ of the stochastic process $\eta(t,\xi)$ is altered through translation map
- Each $x_k(t,\xi) \in x^I(t,\xi)$ as such has a different autocorrelation structure
- This effect might not be intended

- Research paths
  - Iterative schemes to create pre-map of autocorrelation
  - Rank-based autocorrelation schemes opposed to product-moment schemes

Conclusions

- Imprecision is nearly unavoidable in engineering simulations
- Imprecision has large effects on the estimation of the reliability of large-scale structures
- P-boxes provide a natural and intuitive way to handle imprecision
- p-box stochastic processes and fields have been introduced to deal with imprecision in space-time uncertain quantities

Further work

- Deal with autocorrelation structures of realizations of p-box stochastic process